

Euclid's diagrammatic inferences

John Mumma

Division of Logic, Methodology, and
Philosophy of Science in the Suppes
Center for History and Philosophy of
Science, Stanford University

Workshop on diagrammatic
reasoning
March 13th, 2010

PROPOSITION 2.

To place at a given point (as an extremity) a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.

Thus it is required to place at the point A (as an extremity) a straight line equal to the given straight line BC .

From the point A to the point B let the straight line AB be joined;

[Post. 1]

and on it let the equilateral triangle DAB be constructed.

[I. 1]

Let the straight lines AE , BF be produced in a straight line with DA , DB ;

[Post. 2]

with centre B and distance BC let the circle CGH be described;

[Post. 3]

and again, with centre D and distance DG let the circle GKL be described.

[Post. 3]

Then, since the point B is the centre of the circle CGH , BC is equal to BG .

Again, since the point D is the centre of the circle GKL , DL is equal to DG .

And in these DA is equal to DB ;

therefore the remainder AL is equal to the remainder BG .

[C.N. 3]

But BC was also proved equal to BG ;

therefore each of the straight lines AL , BC is equal to BG .

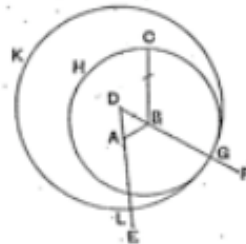
And things which are equal to the same thing are also equal to one another;

[C.N. 1]

therefore AL is also equal to BC .

Therefore at the given point A the straight line AL is placed equal to the given straight line BC .

(Being) what it was required to do.



PROPOSITION 2.

To place at a given point (as an extremity) a straight line equal to a given straight line.

Let A be the given point, and BC the given straight line.

Thus it is required to place at the point A (as an extremity) a straight line equal to the given straight line BC .

From the point A to the point B let the straight line AB be joined;

[Post. 1]

and on it let the equilateral triangle DAB be constructed.

[I. 1]

Let the straight lines AE , BF be produced in a straight line with DA , DB ;

[Post. 2]

with centre B and distance BC let the circle CGH be described;

[Post. 3]

and again, with centre D and distance DG let the circle GKL be described.

[Post. 3]

Then, since the point B is the centre of the circle CGH , BC is equal to BG .

Again, since the point D is the centre of the circle GKL , DL is equal to DG .

And in these DA is equal to DB ;

therefore the remainder AL is equal to the remainder BG .

[C.N. 3]

But BC was also proved equal to BG ;

therefore each of the straight lines AL , BC is equal to BG .

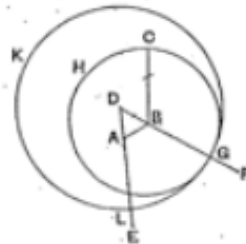
And things which are equal to the same thing are also equal to one another;

[C.N. 1]

therefore AL is also equal to BC .

Therefore at the given point A the straight line AL is placed equal to the given straight line BC .

(Being) what it was required to do.



Goal of talk

Present an account of Euclid's diagrammatic inferences in the *Elements*.

The account is based on a formal system of proof termed **Eu**.

Three parts of talk

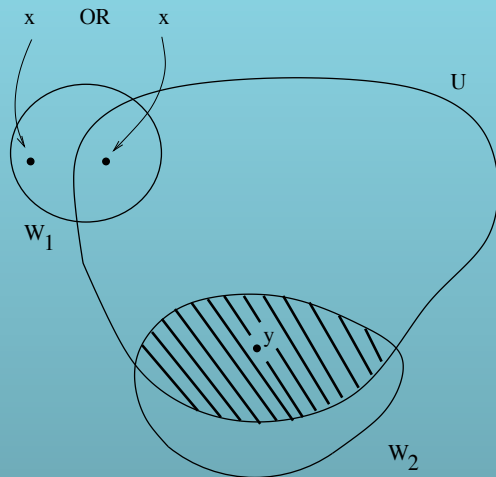
I General remarks on diagrams and mathematical proof

II The structure of **Eu** as a formal system of proof

III The picture **Eu** provides of Euclid's diagrammatic inferences

A common view of diagrams in mathematics

Diagrams are useful for heuristic and pedagogical purposes.



First we show that given a (non-empty) set U of X , and given $x \in X$, there exists a (non-empty) open set V contained in U such that V^ does not contain x .*

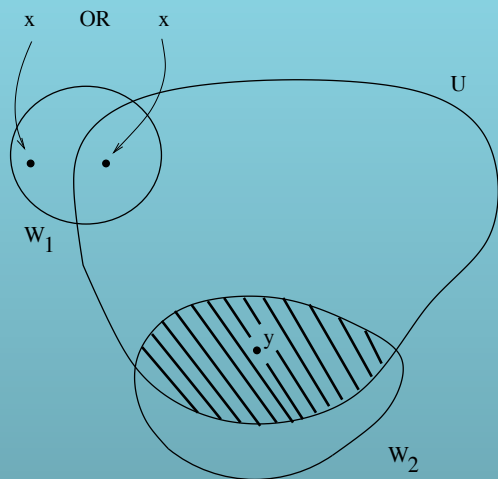
The point x may or may not be in U . But in either case, we can choose a point y in U that is different from x . This is possible if x is in U because x is a limit point of X (so that U must contain a point y that is different from x). And it is possible if x is not in U because U is non empty. Let W_1 and W_2 be disjoint neighborhoods of x and y , respectively; then $V = U \cap W_2$ is the desired open set, whose closure does not contain x . See Figure 16.

Proof from *Topology* by James Munkres

But they ought not to play a role in rigorous proofs.

A common view of diagrams in mathematics

Diagrams are useful for heuristic and pedagogical purposes.



First we show that given a (non-empty) set U of X , and given $x \in X$, there exists a (non-empty) open set V contained in U such that V^ does not contain x .*

The point x may or may not be in U . But in either case, we can choose a point y in U that is different from x . This is possible if x is in U because x is a limit point of X (so that U must contain a point y that is different from x). And it is possible if x is not in U because U is non empty. Let W_1 and W_2 be disjoint neighborhoods of x and y , respectively; then $V = U \cap W_2$ is the desired open set, whose closure does not contain x . See Figure 16.

Proof from *Topology* by James Munkres

The ideal of a mathematically rigorous proof: what is being assumed, and the soundness of the inferences from what is being assumed, ought to be transparent.

If a proof relies on a diagram, this is taken to be compromised.

The case of elementary geometry

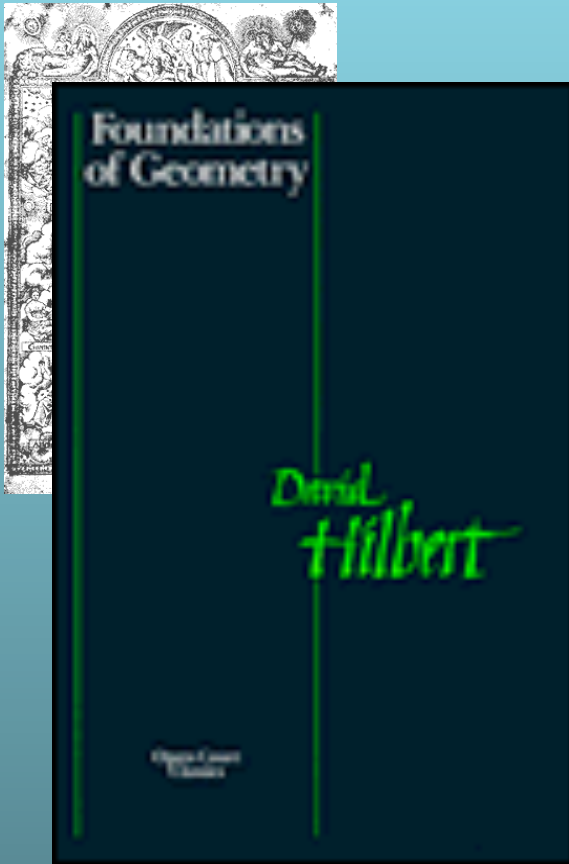
A comparison of ancient and modern treatments of elementary geometry seems to confirm the view.



For more than two thousand years, the canonical foundational text for the subject was Euclid's *Elements*, a work with diagrammatic geometric proofs.

The case of elementary geometry

A comparison of ancient and modern treatments of elementary geometry seems to confirm the view.

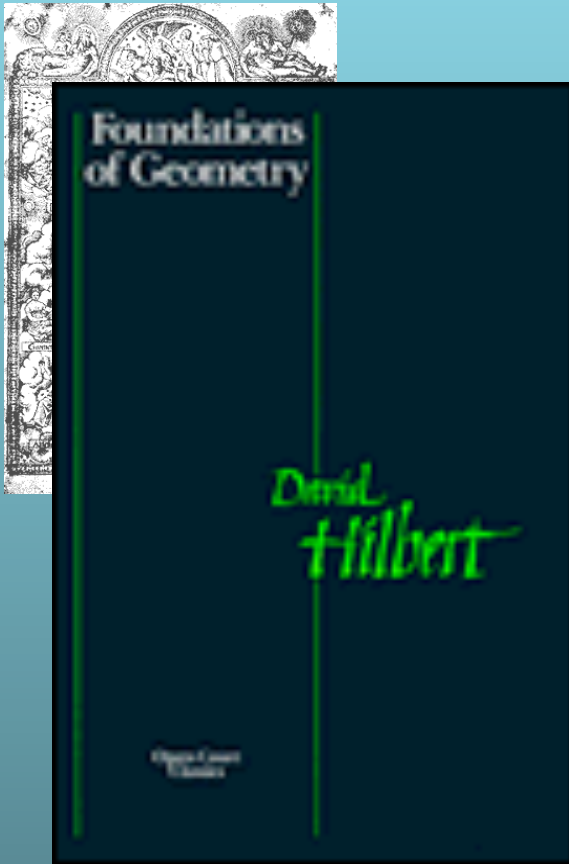


For more than two thousand years, the canonical foundational text for the subject was Euclid's *Elements*, a work with diagrammatic geometric proofs.

This changed roughly a century ago when work appeared that showed that the Euclid's diagrams are dispensable.

The case of elementary geometry

A comparison of ancient and modern treatments of elementary geometry seems to confirm the view.



In the light of this work, the steps where Euclid relies on diagrams seem open-ended and fuzzy.

In a work like Hilbert's *Foundations of Geometry* everything is based on hard and clear axioms and airtight logical deductions from them.

The motivation of Eu

Consequently, Euclid's reliance on diagrams is standardly taken to be a methodological flaw of the *Elements*.

Assumption of view: ideal of proof as exhibited in Hilbert's work the only one.

Specific motivation of **Eu**: exhibit precisely an alternative ideal, whereby there are hard and clear rules for using diagrams in proofs.

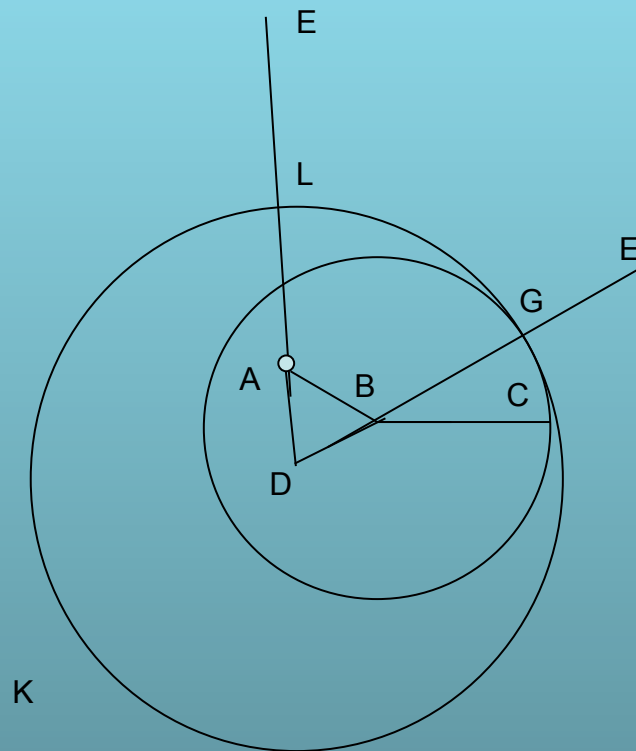


Diagram for proposition 1,2
in the *Elements*

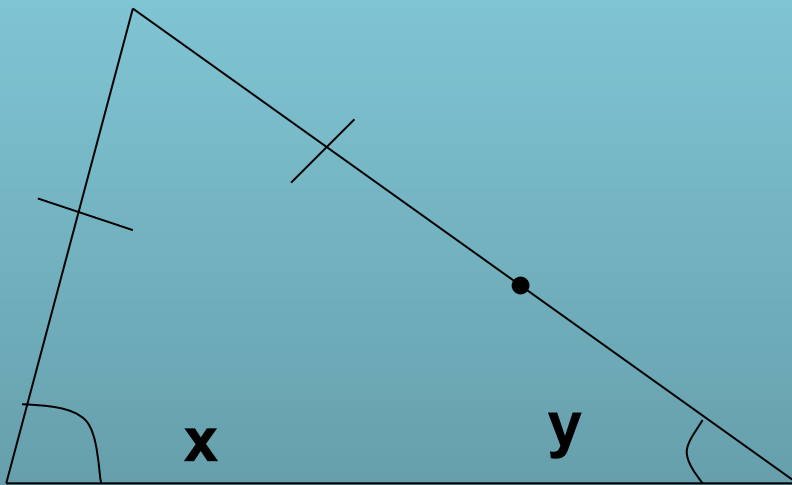
General motivation: question of rigorous mathematical reasoning

Proof from *Elements*

Proposition 18, Book I

In any triangle the angle opposite the greater side is greater.

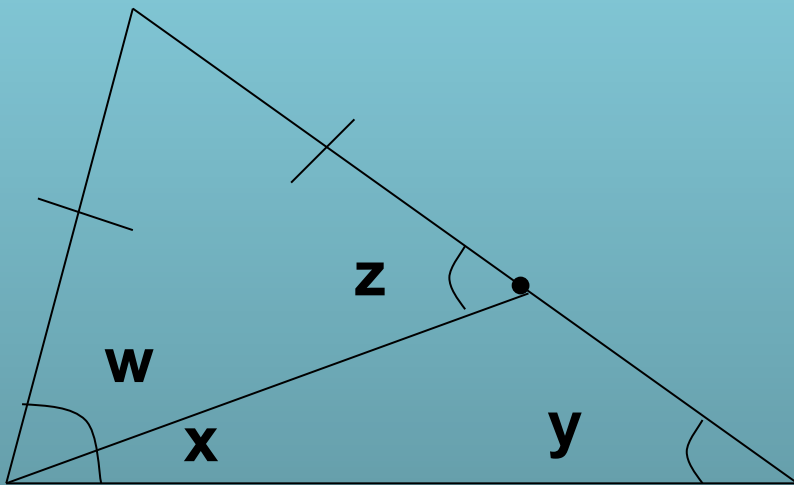
GOAL: Show $x > y$



Proof from *Elements*

Proposition 18, Book I

In any triangle the angle opposite the greater side is greater.



GOAL: Show $x > y$

$z > y$ and $w = z$

(by previous theorems)

So $w > y$

Further, $x > w$

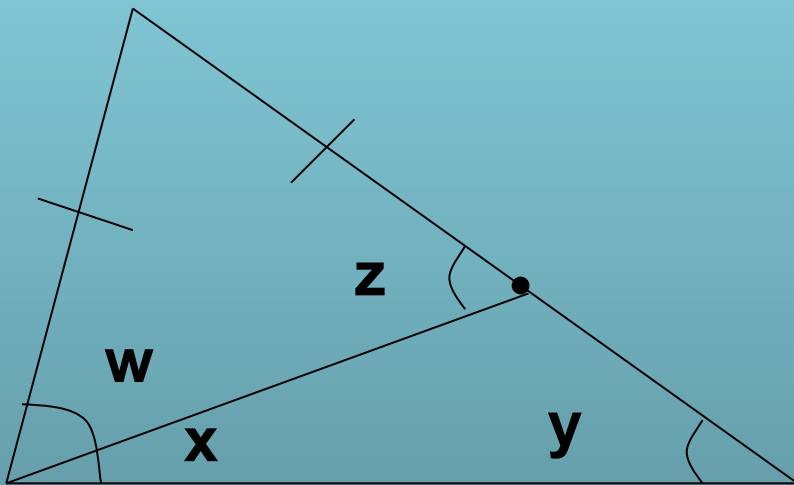
So, $x > y$

QED

Proof from *Elements*

Proposition 18, Book I

In any triangle the angle opposite the greater side is greater.



GOAL: Show $x > y$

$z > y$ and $w = z$

(by previous theorems)

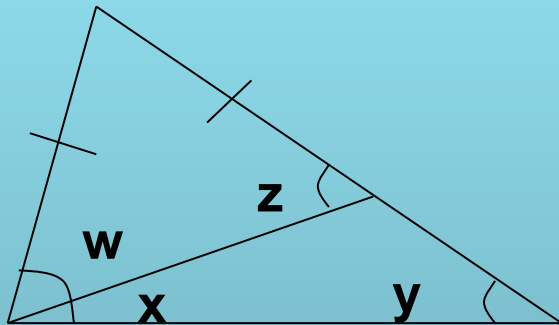
So $w > y$

Further, $x > w$

So, $x > y$

QED

Euclid's use of diagrams



GOAL: Show $x > y$

$z > y$ and $w = z$

(by previous theorems)

So $w > y$

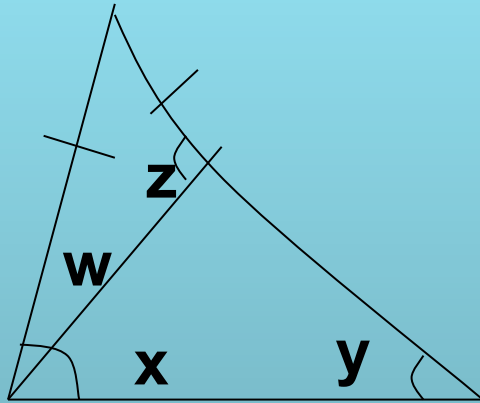
Further, $x > w$

So, $x > y$

QED

Key observation: Euclid's diagrammatic inferences (here and elsewhere) concern only inexact positional relations.

Euclid's use of diagrams



GOAL: Show $x > y$

$z > y$ and $w = z$

(by previous theorems)

So $w > y$

Further, $x > w$

So, $x > y$

QED

Key observation: Euclid's diagrammatic inferences (here and elsewhere) concern only inexact positional relations.

“...geometry is the art of reasoning correctly about figures which are poorly constructed.”

Henri Poincaré

First person to emphasize the philosophical significance of the fact is Ken Manders, in a piece called *The Euclidean Diagram*.

The formal system Eu

What it is composed of:

- Sentential symbols S
- Diagrammatic symbols D
- Rules R for manipulating S and D

The rules are formal in that they depend only on the syntactic form of the symbols. (Like rules from algebra, e.g. $n(x + y) = nx + ny$)

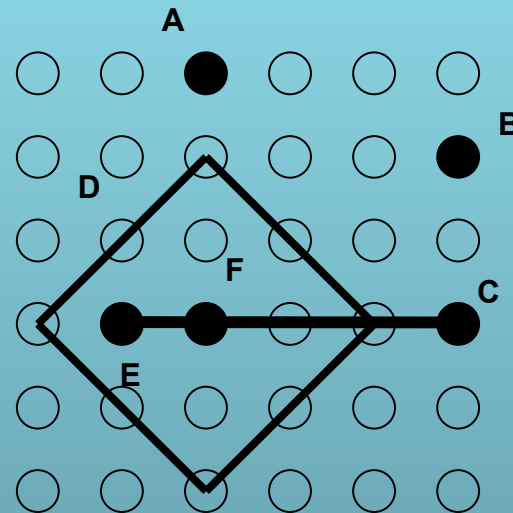
The formal system Eu

What it is composed of:

- Sentential symbols S
- Diagrammatic symbols D
- Rules R for manipulating S and D

The rules are formal in that they depend only on the syntactic form of the symbols. (Like rules from algebra, e.g. $n(x + y) = nx + ny$)

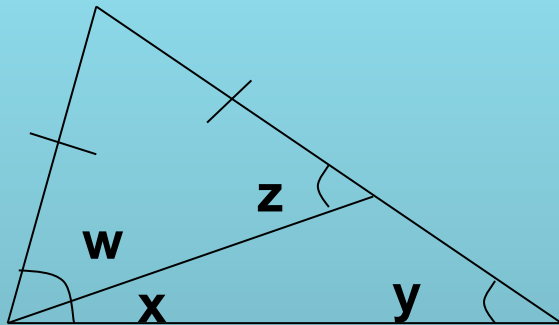
Syntax of Eu diagrams



$n \times n$ arrays for any n

Rules for well-formed diagram specify how array elements can be distinguished as points, lines, and circles.

Geometric proofs in Eu



GOAL: Show $x > y$

$z > y$ and $w = z$

(by previous theorems)

So $w > y$

Further, $x > w$

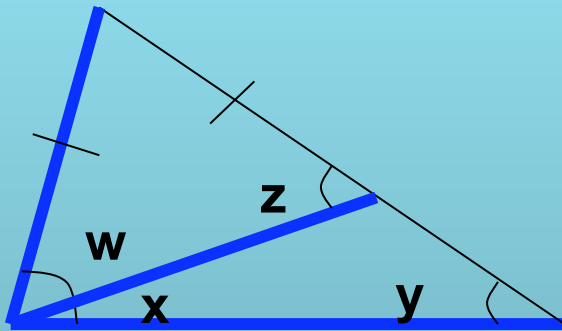
So, $x > y$

QED

Geometric proofs carried out according to the rules of **Eu** have two tracks: a sentential one, and a diagrammatic one.

The role of the sentential track is to record and ground inferences regarding *metric* information.

Geometric proofs in Eu



GOAL: Show $x > y$

$z > y$ and $w = z$

(by previous theorems)

So $w > y$

Further, $x > w$

So, $x > y$

QED

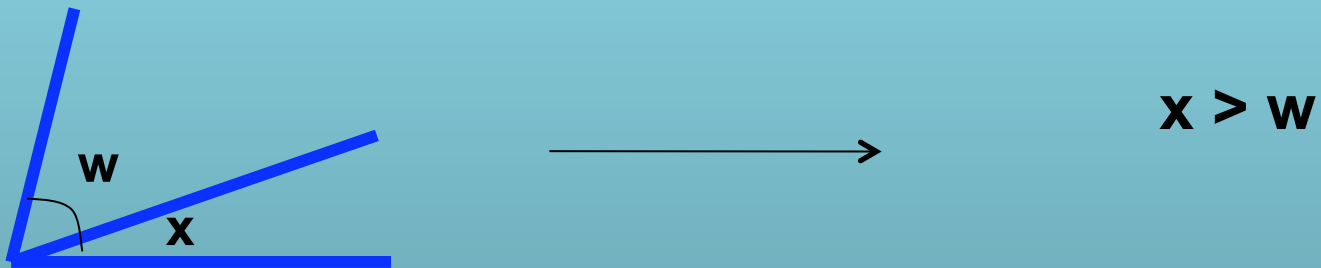
Geometric proofs carried out according to the rules of **Eu** have two tracks: a sentential one, and a diagrammatic one.

The role of the diagrammatic track is to record and ground inferences concerning *non-metric positional* information.

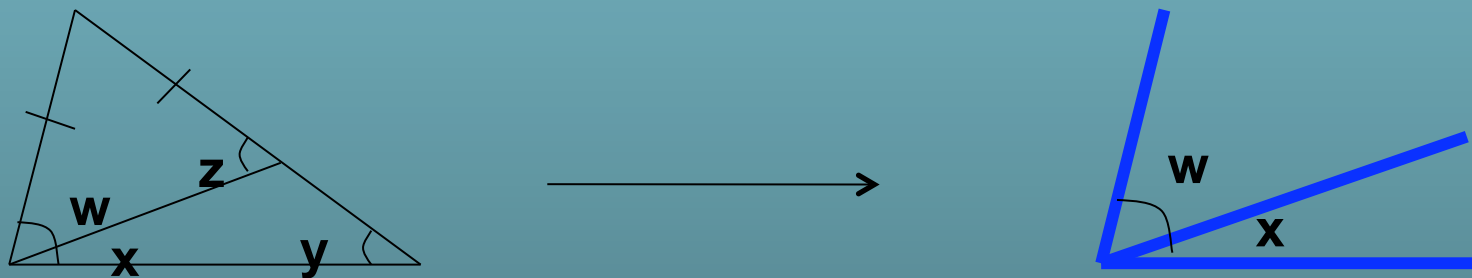
A particular configuration is constructed, and a non-metric positional relation is inferred as general.

Kinds of rules

$z > y, w = z$ \longrightarrow $w > y$
From metric assertions to metric assertions

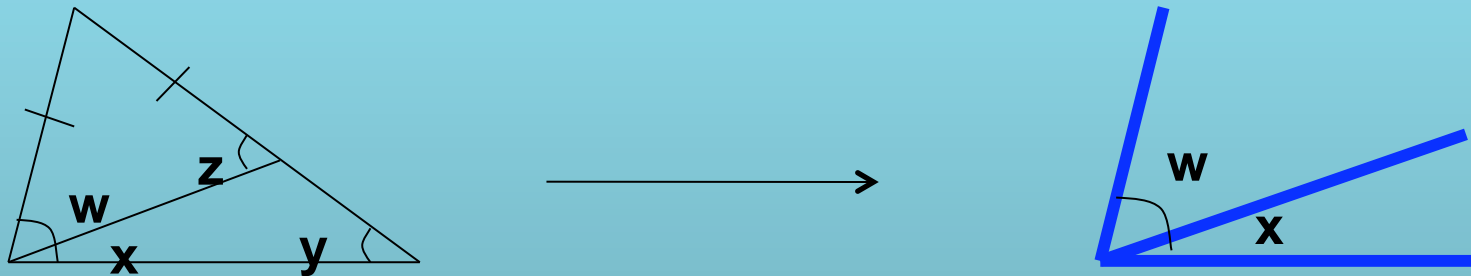


From inferred sub-diagram to metric assertion



From constructed diagram to sub-diagram

Diagrams and generality



From constructed diagram to sub-diagram

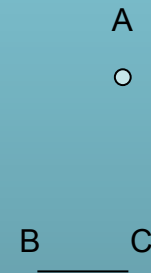
The non-metric positional relations in a diagram can depend on the particular diagram that serves to represent the initial configuration of proof.

What the rules must do: restrict inferences to those that are independent of the particular diagram used.

Example: Proposition I,2

A construction problem: to produce from a given point a segment equal to a given segment.

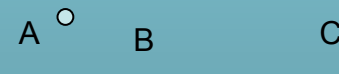
In laying down the initial conditions of the problem in a diagram, one is free to choose any point as the given point.



Example: Proposition I,2

A construction problem: to produce from a given point a segment equal to a given segment.

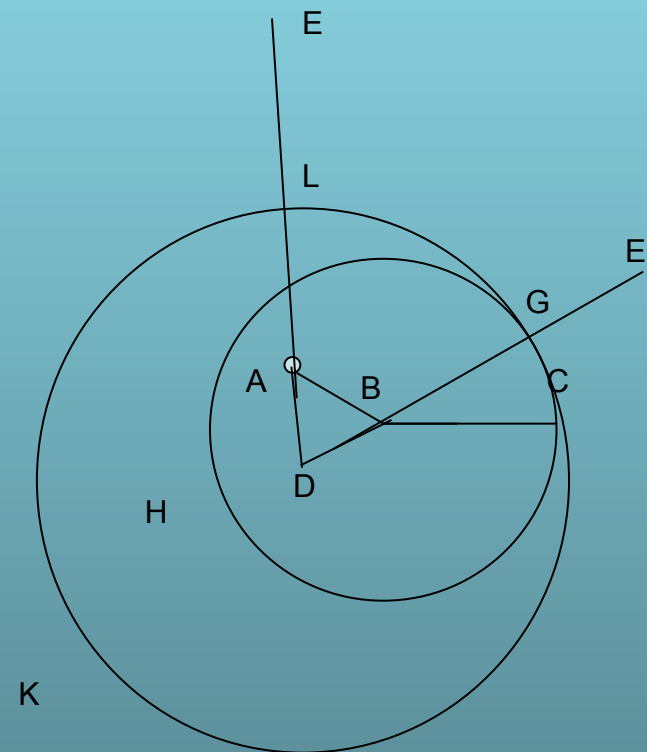
In laying down the initial conditions of the problem in a diagram, one is free to choose any point as the given point.



Example: Proposition I,2

A construction problem: to produce from a given point a segment equal to a given segment.

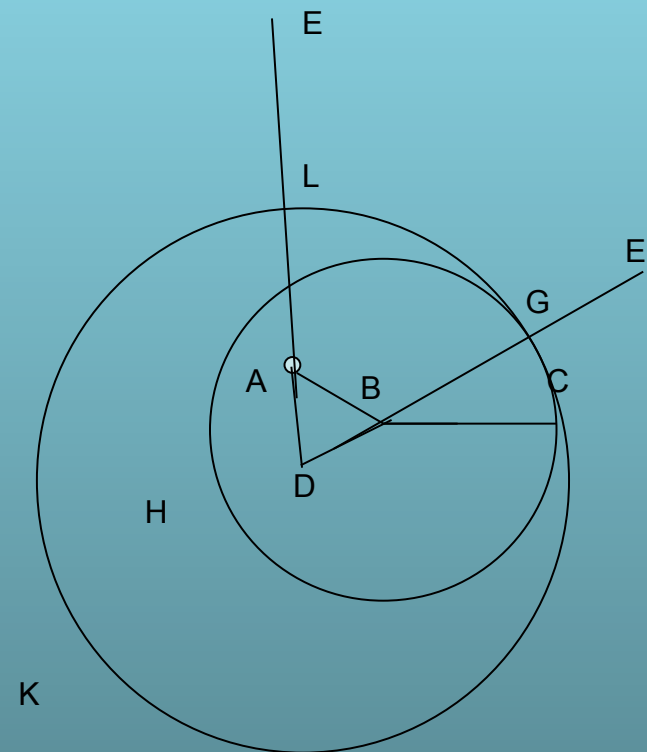
In laying down the initial conditions of the problem in a diagram, one is free to choose any point as the given point.



Example: Proposition I,2

A construction problem: to produce from a given point a segment equal to a given segment.

In laying down the initial conditions of the problem in a diagram, one is free to choose any point as the given point.

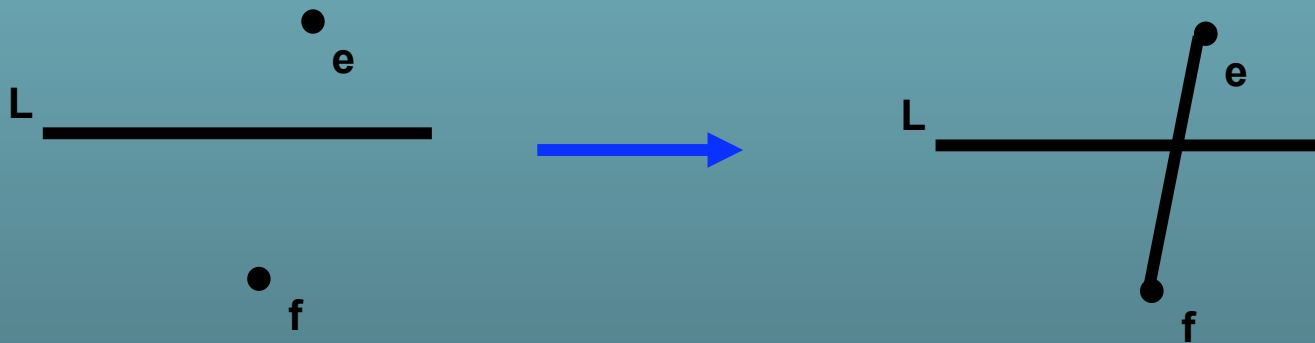


And the diagrammatic proof of the construction is intended to hold for any chosen point.

Diagrams and generality

Method prescribed by **Eu** for isolating what's general
Determine positional invariants of construction by examining each construction step individually.

Eu licenses those inferences such that given the conditions on the position of the construction the manifest position of the constructed object has to hold.



Isolating what's general

Construction for I,2

Let A be a point and BC a line segment.

Join the straight AB from the point A to the point B

Construct the equilateral triangle ABD on segment AB.

Produce AE and BF in line with DA and DB.

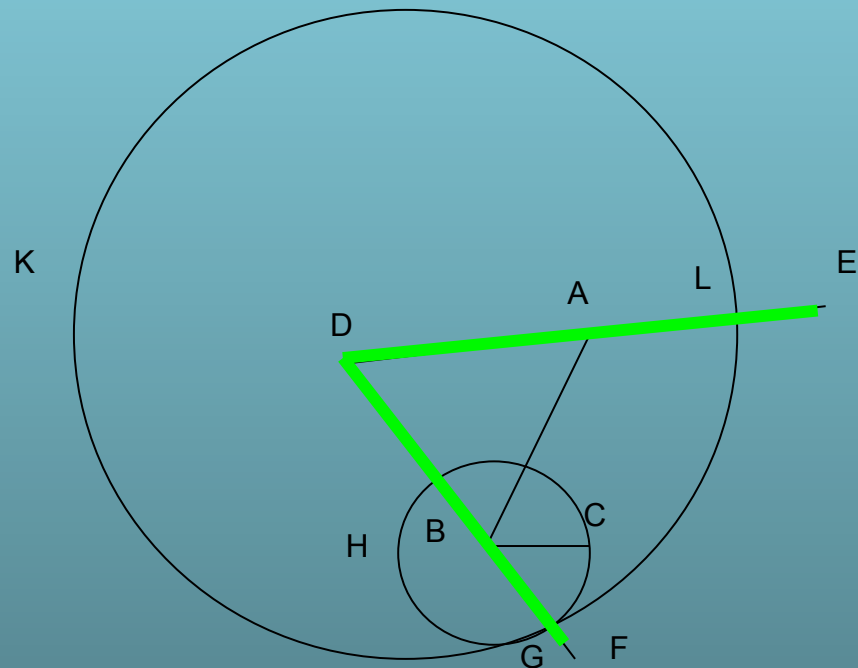
Describe the circle CGH with center B and radius BC.

Describe the circle GKL with center D and radius DG.

Relation needed from
diagram for conclusion:

Position of point B on
segment DG.

Position of point A on
segment DE.



Isolating what's general

Construction for I,2

Let A be a point and BC a line segment.

Join the straight AB from the point A to the point B

Construct the equilateral triangle ABD on segment AB.

Produce AE and BF in line with DA and DB.

Describe the circle CGH with center B and radius BC.

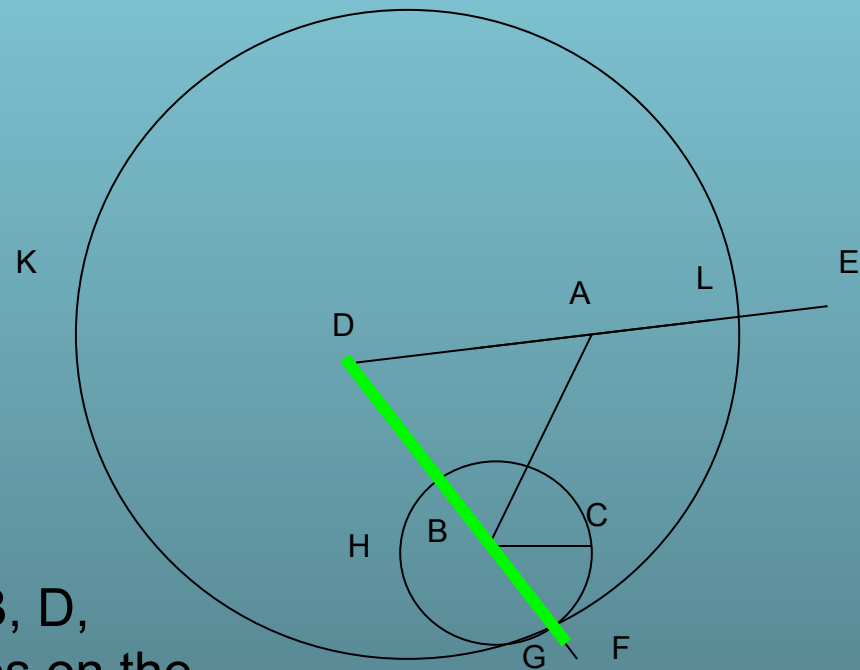
Describe the circle GKL with center D and radius DG.

Concentrating on:

Position of point B on
segment DG.

The question is:

How can we verify that points B, D,
and G exist **in general** and B lies on the
segment DG **in general**?



Isolating what's general

Construction for I,2

Let A be a point and BC a line segment.

Join the straight AB from the point A to the point B

Construct the equilateral triangle ABD on segment AB.

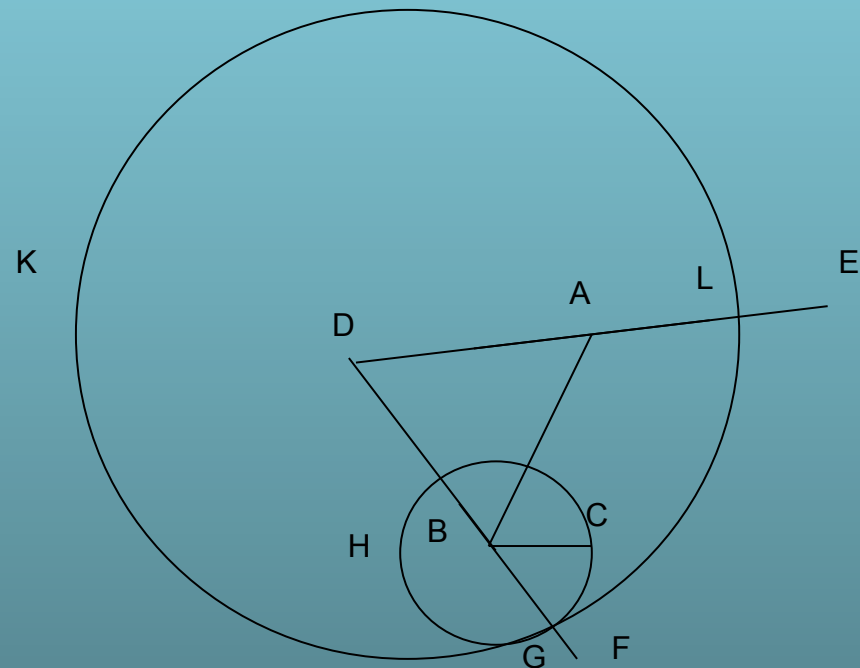
Produce AE and BF in line with DA and DB.

Describe the circle CGH with center B and radius BC.

Describe the circle GKL with center D and radius DG.

Answer:

Inspect diagram in terms of
its construction.



Isolating what's general

Construction for I,2

Let A be a point and BC a line segment.

Join the straight AB from the point A to the point B

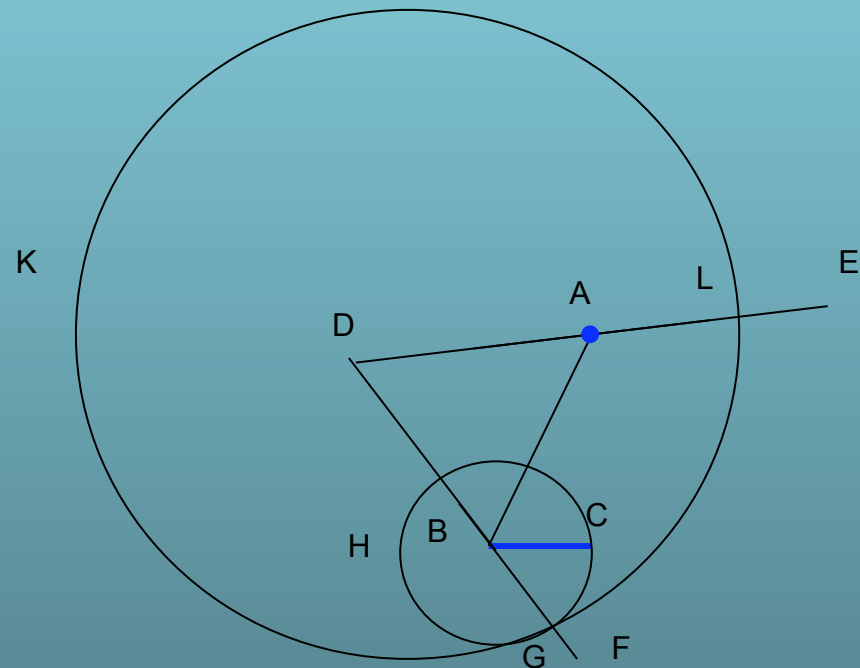
Construct the equilateral triangle ABD on segment AB .

Produce AE and BF in line with DA and DB .

Describe the circle CGH with center B and radius BC .

Describe the circle GKL with center D and radius DG .

A and segment BC exist in general



Isolating what's general

Construction for I,2

Let A be a point and BC a line segment.

Join the straight AB from the point A to the point B

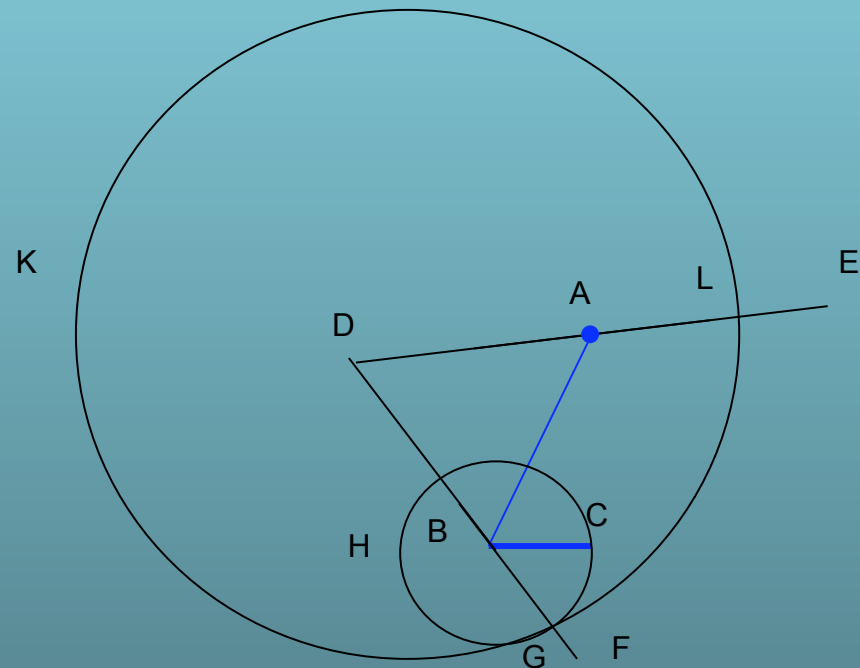
Construct the equilateral triangle ABD on segment AB.

Produce AE and BF in line with DA and DB.

Describe the circle CGH with center B and radius BC.

Describe the circle GKL with center D and radius DG.

A and segment BC exist in general
Segment AB exists in general.



Isolating what's general

Construction for I,2

Let A be a point and BC a line segment.

Join the straight AB from the point A to the point B

Construct the equilateral triangle ABD on segment AB.

Produce AE and BF in line with DA and DB.

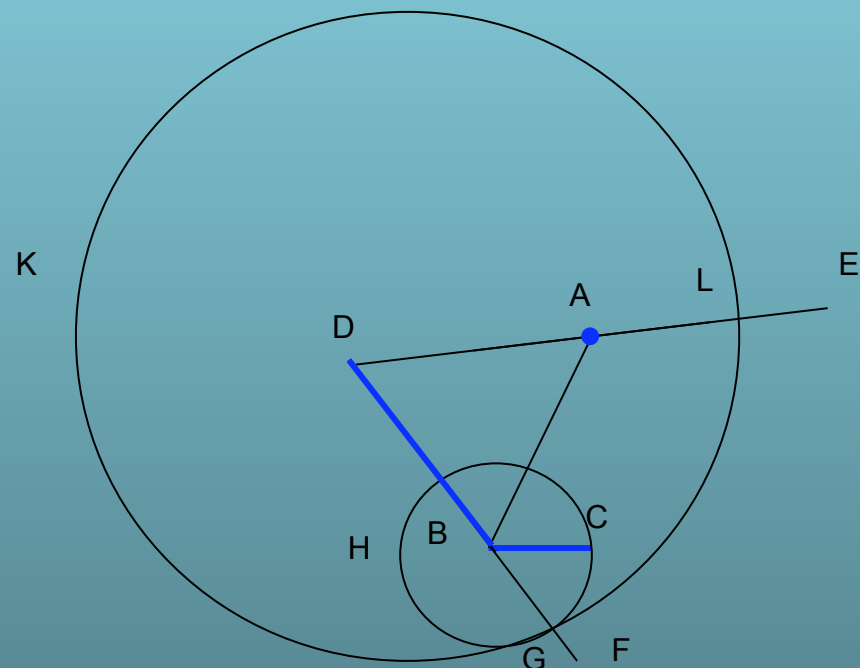
Describe the circle CGH with center B and radius BC.

Describe the circle GKL with center D and radius DG.

A and segment BC exist in general

Segment AB exists in general.

Segment DB exists in general.



Isolating what's general

Construction for I,2

Let A be a point and BC a line segment.

Join the straight AB from the point A to the point B

Construct the equilateral triangle ABD on segment AB.

Produce AE and BF in line with DA and DB.

Describe the circle CGH with center B and radius BC.

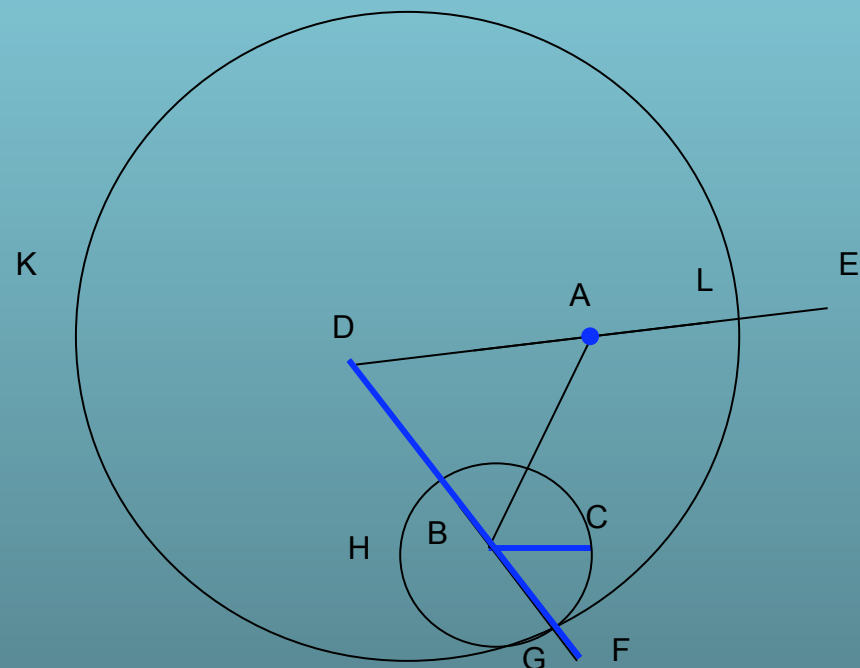
Describe the circle GKL with center D and radius DG.

A and segment BC exist in general.

Segment AB exists in general.

Segment DB exists in general.

Ray DBF exists in general.



Isolating what's general

Construction for I,2

Let A be a point and BC a line segment.

Join the straight AB from the point A to the point B

Construct the equilateral triangle ABD on segment AB .

Produce AE and BF in line with DA and DB .

Describe the circle CGH with center B and radius BC .

Describe the circle GKL with center D and radius DG .

A and segment BC exist in general.

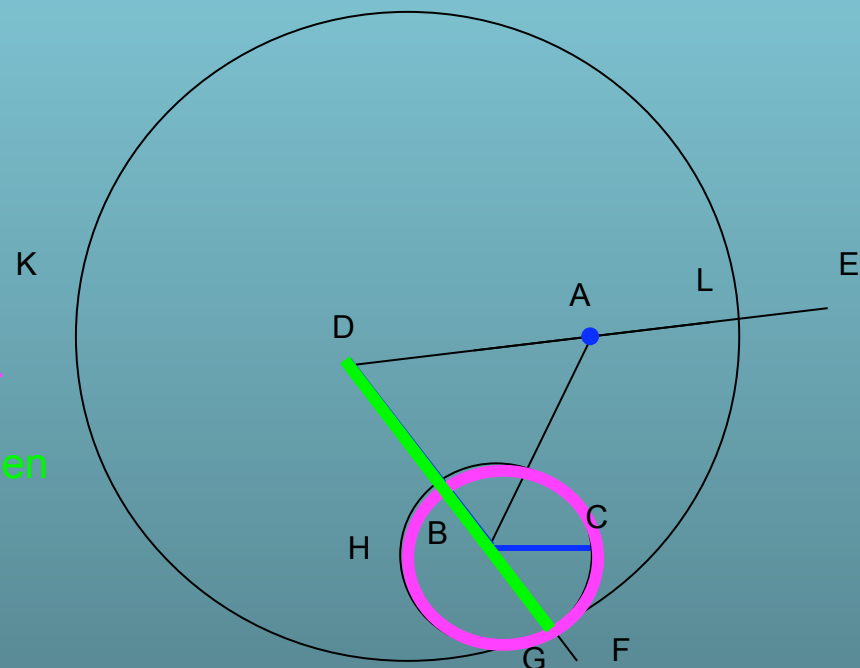
Segment AB exists in general.

Segment DB exists in general.

Ray DBF exists in general.

Circle H with center B exists in general.

Point G exists on line DB , with B between D and G in general.



Form of inferences

$$\frac{R_1(f, x) \quad C(x, y)}{R_2(f, y)}$$

Form of inferences

$R_1(f, x) \quad C(x, y)$

$R_2(f, y)$

f : circle D

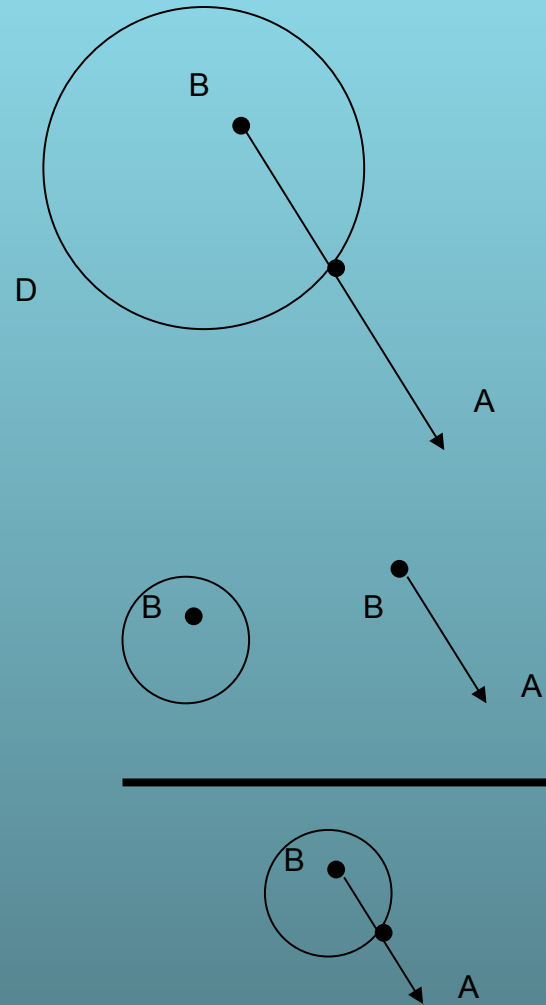
x: point B

y: ray BA

R_1 : position of B inside D

C: construction of BA from B

R_2 : Intersection of D with ray BA



Form of inferences

$$\frac{R_1(f, x) \quad C(x, y)}{\quad}$$

$$R_2(f, y)$$

f : Line L

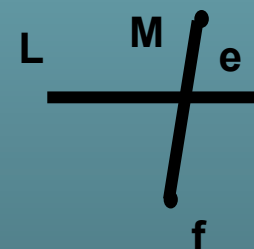
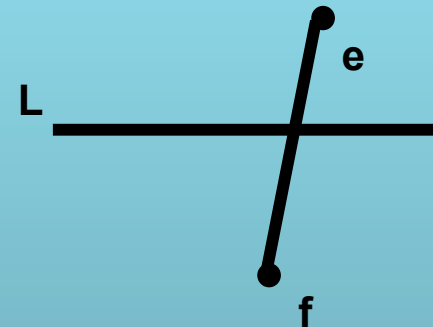
x: points e, f

y: segment M

R_1 : e, f on opposite sides of L

C: construction of M from e and f

R_2 : Intersection of L with segment M



Conclusion

Eu is a formal system, so it fixes a finite collection of these diagrammatic inferences as basic rules.

Though this is artificial, the general form of the rules is less so. A case can be made that inferences of this form are basic to Euclid's geometric method.

By the resulting picture, the method would be *both* formal and intuitive. The idea: geometric intuition can be called upon, but only in formally constrained ways.

Conclusion

Working this idea out means looking at these diagrammatic inferences more closely.

Questions

What (if anything) makes the inferences more trustworthy from a mathematical perspective?

How do the inferences relate (if at all) to our faculties of spatial cognition?