These notes are largely based on [3], with occasional references to other texts. Please drop me an email at `alanaw1` ("at stanford") if you have any comments or discover any errors.

Warm Up...

1. Graphs

   A **graph** is a pair $\mathcal{G} = (V, E)$ where V is a set of **vertices** and E is a set of **edges**. Formally, E is a anti-reflexive symmetric binary relation on V, which means that for $v, w \in V$, $E(v, w) \Rightarrow v \neq w$ and $E(v, w) \Leftrightarrow E(w, v)$.

   ⚽ Euler's seven bridges of Königsberg, Ramsey's subcliques

2. Posets

   A **partially ordered set** or **poset** is a pair $P = (X, \leqslant)$, where X is a set of points and $\leqslant$ is a reflexive, antisymmetric and transitive relation on X. This means that for $x \in X$, $x \leqslant x$ and for $x, y, z \in X$, $x \leqslant y, y \leqslant x \Rightarrow x = y$ and $x \leqslant y, y \leqslant z \Rightarrow x \leqslant z$.

   ⚽ $(\mathbb{Z}, \leqslant)$ under the usual ordering is a partial order. So are set systems $(\mathcal{F}, \subseteq)$ where $\mathcal{F} \subseteq \mathcal{P}(\Omega)$, and many other structures!

3. Groups

   A **group** is a set G endowed with a binary operation $\circ : G \times G \to G$ that is associative and has an identity and inverses with respect to the operation. This means that (1) there exists $e \in G$ such that $e \circ g = g \circ e = g$ for all $g \in G$, and (2) $(\forall g \in G)(\exists h \in G)(g \circ h = h \circ g = e)$.

   ⚽ $GL_2(\mathbb{R})$, the group of invertible 2×2 matrices (the ways to lengthen and rotate a given vector form a set of matrices, and such a set of matrices behave like a group under matrix multiplication $\circ$).

4. Mathematics vs Logic

   The point of these examples was to facilitate the entering into a discussion about mathematical logic. Whereas mathematics *abstracts concepts* from real-world observables, (the project of) logic *starts with meaningless symbols* and *provides meaning* to strings of symbols. Logicians investigate and apply the ideas arising from such a project. Games, models and pretty much everything we discuss today can be viewed as mathematical tools that aid this endeavour.

5. A Golden Rule

   Always revisit this last point, especially if this is your first encounter with mathematical logic. It helps motivate a lot of what is to come below.

Intro to Model Theory (Bits of Chp 4, [3] + Bits of Chp 5, [3])

1. Vocabulary

   A **vocabulary** is a set L of **predicate symbols** $P, Q, R, \ldots$, **function symbols** $f, g, h, \ldots$, and constant symbols $c, d, e, \ldots$. Each vocabulary has an **arity function**

   $$\#_L : L \to \mathbb{N}$$

   telling the arity of each symbol. So $P \in L$ has arity $\#_L(P) = m$ if it is a m-ary predicate symbol, $f \in L$ has arity n if it is a n-ary function symbol. Constant symbols c have arity zero.

   Predicate or function symbols of arity 1 are called **unary** or **monadic**, and those of arity 2 are called **binary**.

   A vocabulary is called unary (binary) if it contains only unary (binary) symbols.

   A vocabulary is called **relational** if it contains no function or constant symbols.

In Chapter 6, we will see the full picture of vocabularies, where **languages** are developed from vocabularies by means of adding **logical symbols** and a systematic procedure of **building formulas**.

2. Models (or Structures)

   A ***L*-structure** or ***L*-model** is a pair $\mathcal{M} = (M, \mathrm{Val}_{\mathcal{M}})$, where $M \neq \emptyset$ is the **universe** (or **domain**) of $\mathcal{M}$, and $\mathrm{Val}_{\mathcal{M}}$ is a function defined on L with the following properties:

   (1) If $R \in L$ is a relation symbol and $\#_L(R) = n$, then $\mathrm{Val}_{\mathcal{M}}(R) \subseteq M^n$.

   (2) If $f \in L$ is a function symbol and $\#_L(f) = n$, then $\mathrm{Val}_{\mathcal{M}}(f) : M^n \to M$.

   (3) If $c \in L$ is a constant symbol, then $\mathrm{Val}_{\mathcal{M}}(c) \in M$.

   Denote the class of all L-structures by $\mathrm{Str}(L)$.

   Usually we write $R^{\mathcal{M}}$, $f^{\mathcal{M}}$ and $c^{\mathcal{M}}$ instead of the notation in (1)-(3) above.

   If the context is clear, we use the notation

   $$\mathcal{M} = (M, R_1^{\mathcal{M}}, \ldots, R_n^{\mathcal{M}}, f_1^{\mathcal{M}}, \ldots, f_m^{\mathcal{M}}, c_1^{\mathcal{M}}, \ldots, c_k^{\mathcal{M}})$$

   for an L-structure $\mathcal{M}$, where $L = \{R_1, \ldots, R_n, f_1, \ldots, f_m, c_1, \ldots, c_k\}$.

3. Examples

   - Let $L = \{E\}$, where E and $=$ are two relation symbols of arity 2. The **first-order language of graphs** is built from equations $x = y$ and edge statements xEy, both called **atomic formulas**, by means of the logical operations $\neg, \wedge, \vee$ and the quantifiers $\exists x$ and $\forall x$, where x ranges over vertices. A model for L is the pair $\mathcal{G} = (V, E)$, where V, the universe, is a set of vertices, and $E = E^{\mathcal{G}}$, interprets the relation symbol E as the edge relation on vertices in V.
   - Let $L = \{\circ\}$, where $\circ$ is a binary function symbol. The first-order language of groups is built from equations, logical operations and quantifiers in a way similar to the way the first-order language of graphs is built. Models $\mathcal{M} = (M, \circ^{\mathcal{M}})$ for L are called **algebras**, which include groups.
   - Let $L = \{\leqslant\}$, where $\leqslant$ is a binary relation symbol. Again, we may build a first-order language of posets. Models $\mathcal{M} = (M, \leqslant^{\mathcal{M}})$ for L are called **relational structures**, and these include partial orders whereby the relation, under $\mathrm{Val}_{\mathcal{M}}$, is antisymmetric, reflexive and transitive.

4. Truth and Satisfiability

   Having some meaningless vocabulary and symbols on one hand and a handful of structures or models that give meaning to symbols on the other, the missing piece is a systematic procedure of verifying truth. We preview this in Chapter 4, and will see the full extent of such a scheme another day (specifically Chapter 6).

   Truth and Satisfiability for Graphs. Let L be the first-order language of graphs as described above. Assignments give values to variables. Thus an assignment in $\mathcal{G}$ is a function s the domain of which is a finite set of variables and the values of which are elements of $\mathcal{G}$.

   We write

   $$\mathcal{G} \models_s \varphi$$

if the formula φ is true in the graph $\mathcal{G}$ when the free variables occurring in φ are interpreted according to s, i.e., a variable x is interpreted as $s(x) \in V$. The assignment $s[a/x]$ is defined as follows:

$$s[a/x](y) = \begin{cases} a & \text{if } y = x, \\ s(y) & \text{if } y \neq x. \end{cases}$$

With this concept the truth condition of quantified formulas can be given easily:

$$\mathcal{G} \models_s \exists x \varphi \iff \text{there is a vertex } v \text{ such that } \mathcal{G} \models_{s[v/x]} \varphi.$$

Such a definition allows us to take *any* formula φ of the language L and any assignment s, and make a claim about whether or not $\mathcal{G} \models_s \varphi$.

Semantic Game. Truth of a sentence in a graph can be thought of as the existence of a winning strategy of **II** in the semantic game. In the beginning **II** holds a pair (φ, s) consisting of a formula φ and an assignment s of the free variables of φ.

(1) If φ is atomic, and s satisfies it in $\mathcal{G}$, then the player who holds (φ, s) wins the game, otherwise the other player wins. (Winning condition)

(2) If $\varphi = \neg\psi$, then the player holding (φ, s) gives (ψ, s) to the other player.

(3) If $\varphi = \psi \wedge \theta$, then the player holding (φ, s) switches to hold (ψ, s) or (θ, s), with the other player deciding which.

(4) If $\varphi = \psi \vee \theta$, then the player who holds (φ, s) switches to hold (ψ, s) or (θ, s), and he or she decides which.

(5) If $\varphi = \forall x\psi$, then the player who holds (φ, s) switches to hold $(\psi, s[v/x])$ for some v, with the other player choosing the v.

(6) If $\varphi = \exists x\psi$, then the player who holds (φ, s) switches to hold $(\psi, s[v/x])$ for some v, and he or she chooses the v.

It follows from the usual definition of satisfiability and truth that $\mathcal{G} \models_s \varphi$ if and only if **II** has a winning strategy in the semantic game.

Ehrenfeucht-Fraïssé Games on Graphs (Chp 4, [3])

1. Motivation: Capturing Isomorphism

   The Ehrenfeucht-Fraïssé Game is played on two graphs, $\mathcal{G}$ and $\mathcal{H}$, by two players, **I** (**Spoiler**, **Abelard**) and **II** (**Duplicator**, **Heloise**). **I** tries to demonstrate a difference between the graphs, while **II** tries to defend the claim that there is no difference. It might seem strange at first: one could simply compare $\mathcal{G}$ and $\mathcal{H}$ by inspection, so why bother playing? The critical observation is that if both graphs are really huge and the game has a limited number of moves, then it could be the case that the graphs are different but **I** cannot demonstrate this given his limited number of moves. More importantly, as we shall see, the game provides a way to check for elementary equivalence, satisfiability of formulas, first-order expressibility and more. *So the game is a tool that aids our logical investigation.*

   When are two graphs different? This leads us to the concept of isomorphism. Two graphs $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V', E')$ are **isomorphic** if there exists a **bijective** map $f : V \to V'$ such that $\{v, w\} \in E \Leftrightarrow \{f(v), f(w)\} \in E'$. Clearly, if two graphs are the same then they are isomorphic (below left). On the other hand, two graphs can be different, but possibly share an identical subgraph (below right).

2. Quantifier Rank

   For a formula φ, its **quantifier rank** $\mathrm{QR}(\varphi)$ is defined inductively as follows:

   (1) $\mathrm{QR}(x = y) = \mathrm{QR}(xEy) = 0$,

   (2) $\mathrm{QR}(\neg\varphi) = \mathrm{QR}(\varphi)$,

   (3) $\mathrm{QR}(\varphi \wedge \psi) = \mathrm{QR}(\varphi \vee \psi) = \max\{\mathrm{QR}(\varphi), \mathrm{QR}(\psi)\}$,

   (4) $\mathrm{QR}(\exists x\varphi) = \mathrm{QR}(\forall x\varphi) = \mathrm{QR}(\varphi) + 1$.

   ⚽ $\mathrm{NI}_n(x_1, \ldots, x_n)$ has quantifier rank 0. $\mathrm{DEG}_{\geqslant n}(x)$ has quantifier rank n. $\mathrm{DEG}_{\geqslant n+1}(x)$ has quantifier rank $n + 1$.

3. Rules of the Game

   The game $\mathrm{EF}_n(\mathcal{G}, \mathcal{H})$ consists of n rounds of play. There are two graphs, $\mathcal{G}$ and $\mathcal{H}$, known to both players. The graphs are on disjoint vertex sets. Each round has two parts, **I**'s move followed by **II**'s. On the ith move **I** chooses a vertex on either graph (his choice) and marks it x_i. Then **II** chooses a vertex on *the other graph* and marks it y_i. A vertex may receive more than one mark.

   Who wins? At the end of the game let $v_0, \ldots, v_{n-1}$ be the vertices of $\mathcal{G}$ marked $*_0, \ldots, *_{n-1}$ (here $* \in \{x, y\}$, depending on whether **I** or **II** placed the mark). Similarly let $v'_0, \ldots, v'_{n-1}$ be the vertices of $\mathcal{H}'$ marked $\square_0, \ldots, \square_{n-1}$ (here $\square \in \{x, y\}$ and takes on whatever $*$ isn't). For **II** to win she must assure that (1) $\{v_i, v_j\} \in E \Leftrightarrow \{v'_i, v'_j\} \in E'$, and (2) $v_i = v_j \Leftrightarrow v'_i = v'_j$. Equivalently, she must assure that $f : V \to V'$ given by $f(v_i) = v'_i$ is a graph isomorphism. If **II** doesn't win then **I** does.

We say $\mathrm{EF}_n(\mathcal{G}, \mathcal{H})$ is a win for **II** if she has a winning strategy, i.e., no matter what **I** does she wins. We say $\mathrm{EF}_n(\mathcal{G}, \mathcal{H})$ is a win for **I** if similarly he has a winning strategy. This is a finite perfect information game with no draws.

4. Some Elementary Results

Proposition 1 (pg. 24, [2]) *Let $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V', E')$ be given. If $n > \min\{|V|, |V'|\}$, then **II** wins if and only if $\mathcal{G}$ and $\mathcal{H}$ are isomorphic.*

Idea is that **II** just needs to duplicate if the graphs are isomorphic, but when they aren't, then **I** can spoil (there are more rounds than number of vertices in one of the graphs).

Proposition 2 (pg. 41, [3]) *Suppose $\mathcal{G} = (V, E)$ and $\mathcal{H} = (V', E')$ are graphs with non-intersecting vertex sets. **II** wins $EF_2(\mathcal{G}, \mathcal{H})$ if and only if the following conditions hold.*

(1) $\mathcal{G}$ has an isolated vertex if and only if $\mathcal{H}$ has an isolated vertex.

(2) $\mathcal{G}$ has an anti-isolated vertex if and only if $\mathcal{H}$ has an anti-isolated vertex.

(3) $\mathcal{G}$ has a vertex which is neither isolated nor anti-isolated if and only if $\mathcal{H}$ does.

Idea is that **I** can spoil by picking the vertex bearing said characteristic, if the condition holds for one graph but not for the other.

Example 4.10, [3] Let $\mathcal{G}$ and $\mathcal{H}$ be empty graphs on $\geqslant n$ vertices. Then **I** wins $\mathrm{EF}_n(\mathcal{G}, \mathcal{H})$ since she simply duplicates without having to worry about adjacencies. So if $\mathcal{G}$ and $\mathcal{H}$ had different number of vertices, **I** couldn't distinguish that feature.

Example 4.11, [3] Let $\mathcal{G}$ and $\mathcal{H}$ be graphs on an even number of vertices, with at least $2n$ vertices each, and both with every vertex of degree exactly 1. (These are **perfect matching graphs**.) Then **II** wins by duplicating. So if $\mathcal{G}$ and $\mathcal{H}$ had different number of vertices and thus different number of edges, **I** couldn't distinguish that feature.

Example 4.12, [3] Let $\mathcal{G}$ and $\mathcal{H}$ be graphs on $\geqslant n$ vertices, and let them be such that each vertex has degree 1 except one vertex (the "center") which has an edge to every other vertex. These are called **star graphs**. **II** wins $\mathrm{EF}_n(\mathcal{G}, \mathcal{H})$. All she has to do is to duplicate! Explicitly, if **I** chooses the center of one graph, she chooses the center of the other graph. So if $\mathcal{G}$ and $\mathcal{H}$ had a different number of vertices (e.g., one has $2n$ and the other $2n + 1$ vertices), or even if one is infinite and the other finite, **I** couldn't distinguish that feature.

We observe a trend. Suppose n is a given natural number. Then, we can construct graphs $\mathcal{G}$ and $\mathcal{H}$ on *large enough* sets of vertices which are different (in the sense of not being isomorphic), but with **II** winning the game $\mathrm{EF}_n(\mathcal{G}, \mathcal{H})$—meaning that any substructures on $\leqslant n$ vertices of both graphs are indistinguishable or isomorphic.

The reason we brought up the last three examples is to prelude a central result that connects the game and elementary equivalence, which we now state and prove.

5. Ehrenfeucht-Fraïssé Games Determine Elementary Equivalence (up to Quantifier Rank)

We prove that if **II** wins $\mathrm{EF}_n(\mathcal{G}, \mathcal{H})$, then sentences of quantifier rank $\leqslant n$ are either true in both or in neither of graphs $\mathcal{G}$ and $\mathcal{H}$. Put differently, logicians often describe models $\mathfrak{A}$ and $\mathcal{B}$ as **elementary equivalent** if $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi$ for all first-order sentences φ. So this statement that we're about to prove is equivalent to $\mathcal{G}$ and $\mathcal{H}$ being elementary equivalent up to a certain threshold on quantifier rank.

Two points: (1) the converse of the statement is also true (to be proved below), (2) this implies that $\mathcal{G}$ and $\mathcal{H}$ can only be distinguished by formulas of quantifier rank $> n$.

6. Statement of Theorem and Proof

Definition 3 *A* position *in $EF_n(\mathcal{G}, \mathcal{H})$ is defined as for general games (Chapter 3, [3]). Call the corresponding set of pairs*

$$f_p = \{(v_0, v_0'), \ldots, (v_{i-1}, v_{i-1}')\}$$

the map *of the position p, or a* position map *of $EF_n(\mathcal{G}, \mathcal{H})$. Unless **II** has already lost, f_p is a function with $dom(f_p) \subseteq V$ and $rng(f_p) \subseteq V'$. Such functions are called* partial mappings $V \to V'$. *If s is an assignment of $\mathcal{G}$ and f a partial mapping $V \to V'$ such that $rng(s) \subseteq dom(f)$, then the equation*

$$(f \circ s)(x) = f(s(x))$$

defines an assignment $f \circ s$ of $\mathcal{G}'$.

Theorem 4 *Suppose $\mathcal{G}$ and $\mathcal{H}$ are two graphs. If **II** has a winning strategy in $EF_n(\mathcal{G}, \mathcal{H})$, then the graphs $\mathcal{G}$ and $\mathcal{H}$ satisfy the same sentences of quantifier rank $\leqslant n$.*

Proof is on pp. 44-47 of [3]. The theorem can be imagined as follows: if **II** can always construct an isomorphism between subgraphs on n points of both $\mathcal{G}$ and $\mathcal{H}$ (so these substructures look the same), then $\mathcal{G}$ and $\mathcal{H}$ have to satisfy the same sentences of quantifier rank $\leqslant n$.

We now state the converse. Its proof applies the following lemma.

Lemma 5 *For every n and for every set $\{x_1, \ldots, x_n\}$ of variables, there are only* finitely many *logically non-equivalent formulas of the first-order language of graphs of quantifier rank $< n$ with the variables $\{x_1, \ldots, x_n\}$.*

Proof by induction. When $n = 0$ the number of formulas is also 0. Assume that there are finitely many—say $f(n)$ of them—logically non-equivalent formulas of quantifier rank $< n$ with free variables $\{x_1, \ldots, x_n\}$. Logical non-equivalence between formulas means the existence of a model for which one formula is true but the other false. Consider any formula ψ of quantifier rank $< n+1$ with free variables $\{x_1, \ldots, x_n, x\}$. These are formed by Boolean combinations of some of the $f(n)$ formulas, appended by a $\exists x$ or $\forall x$ in front of some or all of them. ψ's truth is completely determined by semantic rules and the truth of its constituent formulas of rank $< n$. A truth table for $f(n)$ propositional symbols has $2^{f(n)}$ rows. Each row has value either 0 or 1. So there are at most $2^{2^{f(n)}}$ truth tables. Clearly multiple truth tables correspond to a particular ψ, so the number of logically non-equivalent formulas of quantifier rank $< n+1$ with free variables $\{x_1, \ldots, x_{n+1}\}$ is bounded above by $f(n+1) = 2^{2^{f(n)}}$.

Theorem 6 *Suppose $\mathcal{G}$ and $\mathcal{H}$ are two graphs. If $\mathcal{G}$ and $\mathcal{H}$ satisfy the same sentences of quantifier rank $\leqslant n$, then **II** has a winning strategy in (i.e., she wins) $EF_n(\mathcal{G}, \mathcal{H})$.*

Proof on pg. 48 of [3].

If we combine the two main theorems proved, we obtain the following result, due to Andrzej Ehrenfeucht (b. 1932) and Roland Fraïssé (b. 1920, d. 2008).

Corollary 7 (Ehrenfeucht, Fraïssé) *Suppose $\mathcal{G}$ and $\mathcal{H}$ are two graphs. The following are equivalent.*

(1) $\mathcal{G}$ and $\mathcal{H}$ satisfy the same sentences of quantifier rank $\leqslant n$.

*(2) **II** has a winning strategy / wins $EF_n(\mathcal{G}, \mathcal{H})$.*

Using Games and More to Determine Expressibility (Chp 2, [2])

The Ehrenfeucht-Fraïssé equivalence has numerous applications.

1. Player **I** has "Small Winning Odds"

   There are finitely many (say k) sentences of quantifier rank $\leqslant n$. So there are finitely many (2^k) sets of such sentences. Each set of sentences corresponds to a possible pair of graphs $\mathcal{G}$ and $\mathcal{H}$ such that both agree on all of them but disagree outside of them. When this happens, **I** wins $\mathrm{EF}_n(\mathcal{G}, \mathcal{H})$. So there are only finitely many scenarios in which **I** has a winning strategy in $\mathrm{EF}_n(\mathcal{G}, \mathcal{H})$. Contrast this with the fact that there are infinitely many graphs on any number of vertices and so there are infinitely possible ways of choosing pairs $(\mathcal{G}, \mathcal{H})$ of graphs to play the game on.

2. Expressibility

   Definition 8 *A property P of graphs is called first-order expressible if it can be written as a first-order sentence A—i.e., so that A holds exactly when P holds.*

   Theorem 9 *Suppose that for all n there exists a pair of graphs $(\mathcal{G}, \mathcal{H})$ such that $\mathcal{G}$ has property P but $\mathcal{H}$ doesn't, yet **II** wins $EF_n(\mathcal{G}, \mathcal{H})$. Then P is not first-order expressible.*

   Indeed, if P were first-order expressible by φ, then it has to have finite quantifier rank, say n. But we can find $\mathcal{G}$ and $\mathcal{H}$ such that $\mathcal{G} \models \varphi$ and $\mathcal{H} \not\models \varphi$, whereas **II** wins $\mathrm{EF}_n(\mathcal{G}, \mathcal{H})$. This contradicts the requirement that $\mathcal{G} \models \varphi \Leftrightarrow \mathcal{H} \models \varphi$, as stipulated by EF.

   ⚽ Examples 4.10, 4.11 and 4.12 provide explicit constructions of $\mathcal{G}$ and $\mathcal{H}$ that demonstrate, respectively, that the properties of having an even number of vertices, an even number of edges, and a vertex with infinite degree, are all not first-order expressible.

   ⚽ More examples can be found in both [2] and [3]. Example 4.13 of [3] and Theorem 2.4.1 of [2] both demonstrate how **connectivity** is not first-order expressible. Theorem 2.4.2 of [2] is a simple extension of Theorem 2.4.1 that demonstrates how **two-colourability** is not first-order expressible. (Actually, ***s*-colourability** is not first-order expressible for any $s \geqslant 2$, thanks to a theorem about chromatic numbers and girths of graphs due to Paul Erdős.)

3. Connections with Probability

   In general, what kinds of graph properties are first-order expressible? This is a question logicians care about. Logicians are interested in **theories** (first-order, second-order, monadic, etc.) that have desirable properties, especially with regard to **expressivity**, **decidability** and **complexity** (if decidable). Thus, finding out the limitations of a language with respect to how much it could express is of great interest.

   The goal of this section is to look at how **random structures** can provide us insights into expressivity of the language of graphs. To do that, we need to understand how to think randomly. We summarize the key ideas below, which are due to various people, e.g., Ronald Fagin, Yu. V. Glebskii *et al.*, Haim Gaifman. This topic is very closed tied to the subject of **zero-one laws**, which are studied by such people as Joel Spencer and Phokion Kolaitis (both working in theoretical computer science).

   (1) The Probability Space $G(n, p)$

   (2) Fagin's Theorem

   Theorem 10 (Fagin 1976, GKLT 1969) *Let P be any first-order property of graphs, and $\mu_n(A)$ be the proportion of all labeled graphs on n vertices possessing the property. Then*

   $$\lim_{n\to\infty} \mu_n(A) \in \{0, 1\}.$$

   That is, every first-order expressible property holds either nearly always or nearly never.

⚽ Having more than half ($\geqslant \frac{1}{2}\binom{n}{2}$) the total number of edges is not first-order expressible.

(3) Proof Requires the Łoś-Vaught Test

(4) The Rado Graph

Précis of General Ehrenfeucht-Fraïssé Games (Chp 5, [3])

We have seen EF Games with n rounds. What happens if there are ω many rounds? What do we know about $\mathrm{EF}_\omega(\mathcal{M},\mathcal{N})$, where $\mathcal{M}$ and $\mathcal{N}$ are general structures, not just graphs?

1. Structural Isomorphism
2. Back-and-Forth Sets
3. Existence of Back-and-Forth Set $\Longleftrightarrow$ **II** wins $\mathrm{EF}_\omega(\mathcal{M},\mathcal{N})$

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