Topological Semantics for Provability Logics

An Overview

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Logic Seminar

Outline

1. Review of Provability Logic **GL**
2. Polymodal Provability Logic **GLP**
3. The Closed Fragment of **GLP**
4. Ignatiev's Model and the Canonical Model
5. Topological Completeness using ϵ_0
6. Topological Completeness of **GLB**
7. More recent developments

$$\varphi \quad ::= \quad P \quad | \quad \varphi \wedge \varphi \quad | \quad \neg\varphi \quad | \quad \Box\varphi$$

$$\Box\varphi \quad \approx \quad \text{'It is provable that } \varphi\text{'}$$

Let $\mathcal{L}$ be an arithmetical language and T an $\mathcal{L}$-theory with an encoding of a (standard) provability predicate $Prov(\cdot)$, so that

$$\mathcal{N} \models Prov(\ulcorner A \urcorner) \quad \text{iff} \quad T \vdash A\,.$$

A **realization** is a function from modal to arithmetical formulas:

$$(\varphi \wedge \psi)^* = \varphi^* \wedge \psi^* \quad (\neg\varphi)^* = \neg\varphi^* \quad (\Box\varphi)^* = Prov(\ulcorner \varphi^* \urcorner)\,.$$

Theorem (Solovay 1976)

φ is **GL**-valid, if and only if it is PA-provable under all realizations.

Axioms and rules of **GL**:

1. *Modus ponens* ;
2. Propositional tautologies ;
3. Necessitation: $\frac{\varphi}{\Box\varphi}$;
4. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
5. $\Box(\Box\varphi \to \varphi) \to \Box\varphi$.

Theorem (Segerberg 1971)

GL is sound and complete w.r.t. finite trees.

Consider an increasing sequence of stronger provability predicates:

$$Prov_0 \quad Prov_1 \quad Prov_2 \quad Prov_3 \quad \ldots$$

where $Prov_n(\ulcorner A \urcorner)$ means A is provable from T together with all true Π_n sentences of $\mathcal{L}$ (cf. ω-provable).

$$\varphi \quad ::= \quad P \mid \varphi \wedge \varphi \mid \neg\varphi \mid [0]\varphi \mid [1]\varphi \mid [2]\varphi \mid \ldots$$

Theorem (Japaridze 1985)

The logic **GLP** is arithmetically complete:

1. **GL** for each modality $[n]$;
2. $[n]\varphi \to [n+1]\varphi$;
3. $\langle n \rangle \varphi \to [n+1]\langle n \rangle \varphi$.

Applications in Ordinal Analysis

- The closed fragment $\mathbf{GLP}^0$ has been used in ordinal analysis.
- The L-T algebra for $\mathbf{GLP}^0$ possess a natural ordering:

$$[\varphi] <_0 [\psi] \quad \text{iff} \quad \vdash_{\mathbf{GLP}} \psi \to \langle 0 \rangle \varphi \ .$$

- It turns out $<_0$ has order-type ϵ_0, the 'proof-theoretic ordinal' for PA. (Recall ϵ_0 is the least fixed point of ω^α.)
- Beklemishev [2004] showed how induction up to ϵ_0, with this algebra providing a **notation system**, can be used to prove consistency of PA.

Frame Incompleteness

Consider a relational frame $\mathbb{F} = (W, \{R_n\}_{n<\omega})$:

(a) $\mathbb{F} \vDash [n]\varphi \to [n+1]\varphi$, iff $R_{n+1} \subseteq R_n$;

(b) $\mathbb{F} \vDash \langle n\rangle\varphi \to [n+1]\langle n\rangle\varphi$, iff $R_n^{-1}(X)$ is R_{n+1}-upward closed ;

Suppose $\mathbb{F} \vDash$ **GLP** and $R_1 \neq \emptyset$, e.g., xR_1y. Then by (a), xR_0y, and by (b), yR_0y, contradicting converse-well-foundedness of R_0.

There is no non-trivial frame semantics for GLP.

This motivates the search for other classes of models.

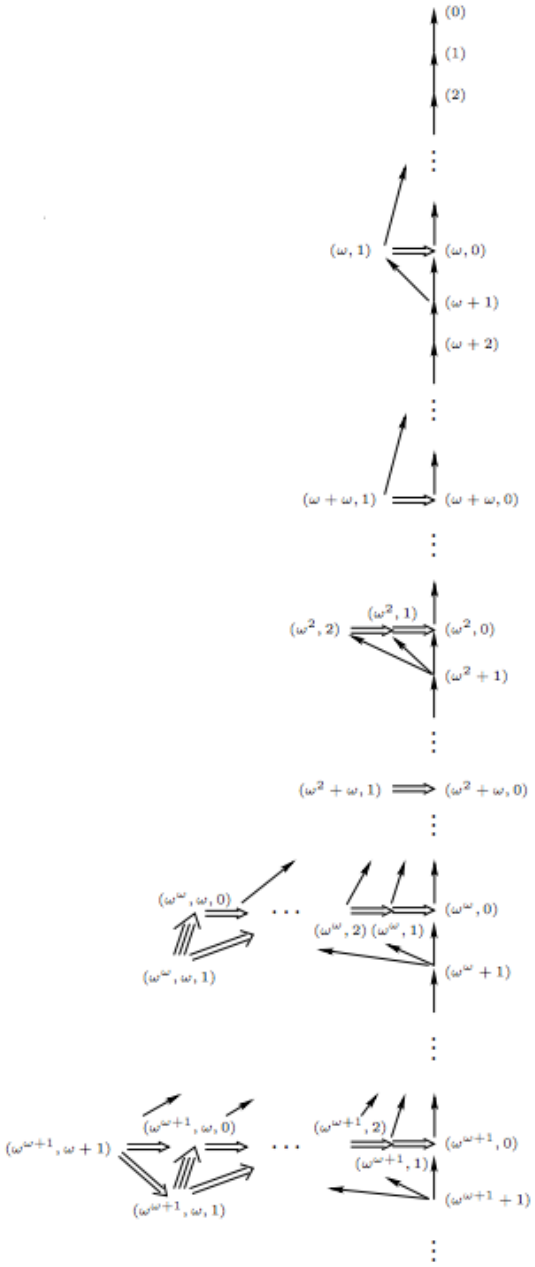
- For $\mathbf{GLP}^0$, Ignatiev [1993] defined a single *universal frame*.
- It is in fact based on ϵ_0.
- Recall the Cantor Normal Form of an ordinal κ (where $\lambda_i \geq \lambda_j$ for $i > j$):

$$\omega^{\lambda_k} + \cdots + \omega^{\lambda_0} \ .$$

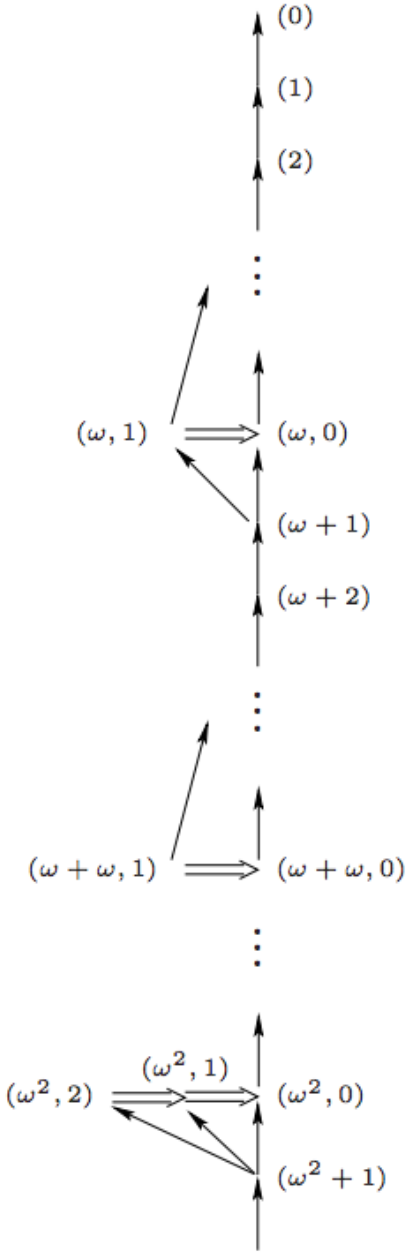
  Define $l(\kappa) = \lambda_0$.
- Let Ω be the set of ω-sequences $(\alpha_1, \alpha_2, \dots)$, with $\alpha_i < \epsilon_0$.
- Ignatiev's frame $\mathcal{U} = (U, \{R_n\}_{n<\omega})$ is defined:

$$\begin{array}{rcl} U & = & \{\alpha \in \Omega : \alpha_{n+1} \leq l(\alpha_n)\} \\ \alpha R_n \beta & \text{iff} & \alpha_n > \beta_n \text{ and } \alpha_m = \beta_m \text{ for } m < n \ . \end{array}$$

Polymodal Provability Logic



Polymodal Provability Logic



- How does $\mathcal{U}$ relate to the canonical model of $\mathbf{GLP}^0$?
- In Icard [2011] this question was answered using tools from topology and the theory of general descriptive frames.

$$
\begin{tikzcd}
(\mathbb{C}^0, C) \arrow[r, shift left, "algebra"] & \mathfrak{T}^0_{\mathcal{L}} \arrow[l, shift left, "u.f.f."] \\
(\mathcal{U}, A) \arrow[u, squiggly, "?"] \arrow[ur, "algebra"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \mathcal{U} & & \\
 & \mathcal{U}^{+} \arrow[u, "0"] & & \\
 & \mathcal{U}^{++} \arrow[u, "1"] \arrow[uu, bend right=60, "0"'] \arrow[uu, bend right=30, "0"'] & & \widehat{\epsilon_0} \arrow[ll, "2"'] \arrow[ull, "1"'] \arrow[uull, "0"'] \arrow[dl, "3"] \\
 & \mathcal{U}^{+++} \arrow[u, "2"] \arrow[uu, bend right=60, "0"'] \arrow[uu, bend right=30, "1"'] & & \ddots \\
 & \vdots \arrow[u, "3"] & &
\end{tikzcd}
$$

- Topological semantics of modal logic goes back to McKinsey and Tarski [1944], who interpreted $\Diamond$ as topological closure.
- This is inappropriate for provability logic, since $X \subseteq cl(X)$, i.e., $\Box\varphi \to \varphi$ is valid on any topological space.
- Instead, we can interpret $\Diamond$ as derivative. Analogous to McKinsey & Tarski's result on **S4**, Esakia showed that the class of all spaces is axiomatized by **wK4**, replacing **4** with:

$$\Box\varphi \wedge \varphi \to \Box\Box\varphi \ .$$

- Esakia also showed **K4** axiomatizes the T_d-spaces.

- The most important result from Esakia [1981] is completeness of **GL** w.r.t. the *scattered spaces*:

  Every subspace has an isolated point.

- Löb's axiom: $\Diamond\varphi \to \Diamond(\varphi \wedge \neg\Diamond\varphi)$. Topologically speaking:

$$d(X) = d\big(X \setminus d(X)\big) \ .$$

- A canonical example of a scattered space is any ordinal with the interval topology (sub-basis of open rays).

Theorem (Abashidze 1985; Blass 1990)

GL is complete with respect to ω^ω with interval topology.

- **GLP**^{0} is also complete w.r.t. a polytopological space defined on ϵ_0.
- Infinitely many finer and finer topologies on ϵ_0:
  - τ_0 is the 'left topology' ;
  - τ_1 is the interval topology (a.k.a. 'order topology') ;
  - τ_2 adds to τ_1 subbasic open sets $\{\alpha : l(\alpha) > \beta\}$
- In general, τ_{n+1} adds to τ_n subbasic sets

$$\{\alpha : l^n(\alpha) > \beta\}.$$

Theorem (Icard 2011)

GLP^0 is complete with respect to $(\epsilon_0, \{\tau_n\}_{n<\omega})$.

- In very recent work, Fernández-Duque and Joosten [2013] generalize the previous theorem to arbitrary ordinals.
- $\mathcal{L}^0_\omega$ is a special case of $\mathcal{L}^0_\Lambda$, and the 'logarithm' function l must be able to iterate into the transfinite.
- They generalize Ignatiev's 'universal frame' for $\mathcal{L}^0_\Lambda$, and by eliminating 'non-root' points and generalizing the topologies in the previous slide, they obtain topological completeness.

Theorem (Fernández-Duque and Joosten 2013)

$\mathbf{GLP}^0_\Lambda$ is complete w.r.t. frame and topological semantics.

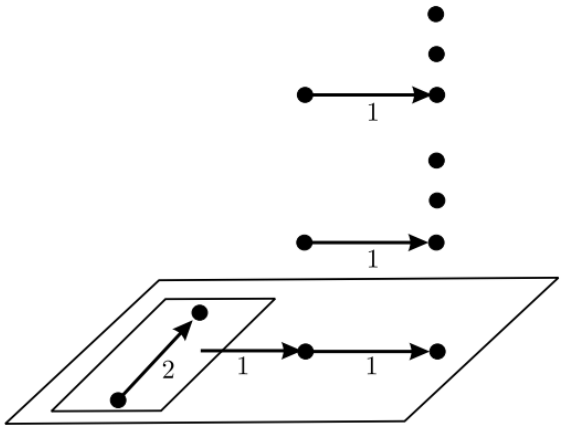
Definition

A polytopological space $(\mathcal{X}, \{\tau_n\}_{n<\omega})$ is a **GLP**-space if, for all n:

- τ_n is scattered ;
- $\tau_n \subseteq \tau_{n+1}$;
- $d_n(X) \in \tau_{n+1}$.

Are there non-trivial GLP-spaces?

- Beklemishev et al. [2010] proved topological completeness of **GLB** w.r.t. **GLP**-spaces, using relational semantics established in Beklemishev [2010].



- τ_0 is the upset topology (standard **GL** model).
- τ_1 is the interval topology.

Ordinal Models

Fact (Beklemishev et al. 2010)

If $(\mathcal{X}, \tau_0, \tau_1)$ is a **GLB**-space and τ_0 is first-countable and Hausdorff, then τ_1 must be discrete.

Fact (Beklemishev 2011)

If $\Omega \geq \aleph_\omega$ and $V = L$, then **GLB** is complete w.r.t. (Ω, τ_0, τ_1), where τ_0 is interval topology and τ_1 is *club topology*.

- If $cf(\alpha) \leq \aleph_0$, then α is isolated ;
- If $cf(\alpha) > \aleph_0$, then any neighborhood of α contains a club.
- Blass [1990] showed incompleteness of **GL** w.r.t. τ_1 is equiconsistent with existence of a weakly Mahlo cardinal.

- Even more recently Beklemishev and Gabelaia [2013] have settled a number of questions.
- They note it is independent of ZFC whether τ_2 can be non-trivial on an *ordinal space*, but show there are assumptions that guarantee all τ_n are non-discrete.
- The question of whether there are set-theoretic assumptions that would give ordinal completeness of **GLP** is open.
- Beklemishev and Gabelaia [2013] do show topological completeness of **GLP** w.r.t. general **GLP**-spaces, but (in their own words) "it is not an example of a *natural* **GLP**-space."

Summary

- The study of topological models of provability logic combines work in proof theory, topology, algebra, set theory, and modal logic, exposing some mathematically interesting structures.
- It also promises a better grip on the polymodal provability logic **GLP**, providing a simple non-arithmetical interpretation.
- Current work in Moscow and Barcelona is focused on using these models to extend Beklemishev's original ordinal analysis of PA to stronger theories and larger ordinals.

Thanks for your attention!

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