Topological Semantics for Provability Logics

An Overview

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Logic Seminar
Outline

1. Review of Provability Logic GL
2. Polymodal Provability Logic GLP
3. The Closed Fragment of GLP
4. Ignatiev’s Model and the Canonical Model
5. Topological Completeness using $\varepsilon_0$
6. Topological Completeness of GLB
7. More recent developments
Modal Logic of Provability

\[ \varphi \ ::= \ P \mid \varphi \land \varphi \mid \neg \varphi \mid \Box \varphi \]

\( \Box \varphi \approx \text{‘It is provable that } \varphi \text{’} \)

Let \( \mathcal{L} \) be an arithmetical language and \( T \) an \( \mathcal{L} \)-theory with an encoding of a (standard) provability predicate \( \text{Prov}(\cdot) \), so that

\[ \mathcal{N} \models \text{Prov}(\Box A) \iff T \vdash A. \]

A realization is a function from modal to arithmetical formulas:

\[ (\varphi \land \psi)^* = \varphi^* \land \psi^* \quad (\neg \varphi)^* = \neg \varphi^* \quad (\Box \varphi)^* = \text{Prov}(\Box \varphi^*) . \]
Theorem (Solovay 1976)

$\varphi$ is $\text{GL}$-valid, if and only if it is PA-provable under all realizations.

Axioms and rules of $\text{GL}$:

1. *Modus ponens* ;
2. Propositional tautologies ;
3. Necessitation: $\frac{\varphi}{\Box \varphi}$ ;
4. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi)$ ;
5. $\Box(\Box \varphi \rightarrow \varphi) \rightarrow \Box \varphi$ .

Theorem (Segerberg 1971)

$\text{GL}$ is sound and complete w.r.t. finite trees.
Consider an increasing sequence of stronger provability predicates:

\[ \text{Prov}_0 \quad \text{Prov}_1 \quad \text{Prov}_2 \quad \text{Prov}_3 \quad \ldots \]

where \( \text{Prov}_n(\Box A) \) means \( A \) is provable from \( T \) together with all true \( \Pi_n \) sentences of \( \mathcal{L} \) (cf. \( \omega \)-provable).

\[ \varphi ::= P \mid \varphi \land \varphi \mid \neg \varphi \mid [0] \varphi \mid [1] \varphi \mid [2] \varphi \mid \ldots \]

**Theorem (Japaridze 1985)**

The logic **GLP** is arithmetically complete:

1. **GL** for each modality \( [n] \);
2. \( [n] \varphi \rightarrow [n + 1] \varphi \);
3. \( \langle n \rangle \varphi \rightarrow [n + 1] \langle n \rangle \varphi \).
Applications in Ordinal Analysis

- The closed fragment $\text{GLP}^0$ has been used in ordinal analysis.
- The L-T algebra for $\text{GLP}^0$ possess a natural ordering:

  \[
  [\varphi] <_0 [\psi] \iff \vdash_{\text{GLP}} \psi \rightarrow \langle 0 \rangle \varphi .
  \]

- It turns out $<_0$ has order-type $\epsilon_0$, the ‘proof-theoretic ordinal’ for PA. (Recall $\epsilon_0$ is the least fixed point of $\omega^\alpha$.)
- Beklemishev [2004] showed how induction up to $\epsilon_0$, with this algebra providing a notation system, can be used to prove consistency of PA.
Frame Incompleteness

Consider a relational frame $F = (W, \{ R_n \}_{n<\omega})$:

(a) $F \vdash [n] \varphi \rightarrow [n+1] \varphi$, iff $R_{n+1} \subseteq R_n$;

(b) $F \vdash \langle n \rangle \varphi \rightarrow [n+1] \langle n \rangle \varphi$, iff $R_{n}^{-1}(X)$ is $R_{n+1}$-upward closed;

Suppose $F \models \text{GLP}$ and $R_1 \neq \emptyset$, e.g., $xR_1 y$. Then by (a), $xR_0 y$, and by (b), $yR_0 y$, contradicting converse-well-foundedness of $R_0$.

There is no non-trivial frame semantics for GLP.

This motivates the search for other classes of models.
For $\mathbf{GLP}^0$, Ignatiev [1993] defined a single *universal frame*.

It is in fact based on $\epsilon_0$.

Recall the Cantor Normal Form of an ordinal $\kappa$ (where $\lambda_i \geq \lambda_j$ for $i > j$):

$$\omega^{\lambda_k} + \cdots + \omega^{\lambda_0}.$$  

Define $l(\kappa) = \lambda_0$.

Let $\Omega$ be the set of $\omega$-sequences $(\alpha_1, \alpha_2, \ldots)$, with $\alpha_i < \epsilon_0$.

Ignatiev’s frame $\mathcal{U} = (U, \{R_n\}_{n<\omega})$ is defined:

$$U = \{ \alpha \in \Omega : \alpha_{n+1} \leq l(\alpha_n) \}$$

$\alpha R_n \beta$ iff $\alpha_n > \beta_n$ and $\alpha_m = \beta_m$ for $m < n$.  

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Polymodal Provability Logic

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How does $\mathcal{U}$ relate to the canonical model of $\text{GLP}^0$?

In Icard [2011] this question was answered using tools from topology and the theory of general descriptive frames.
Topological semantics of modal logic goes back to McKinsey and Tarski [1944], who interpreted $\Diamond$ as topological closure.

This is inappropriate for provability logic, since $X \subseteq cl(X)$, i.e., $\Box \varphi \rightarrow \varphi$ is valid on any topological space.

Instead, we can interpret $\Diamond$ as derivative. Analogous to McKinsey & Tarski’s result on $\mathbf{S4}$, Esakia showed that the class of all spaces is axiomatized by $\mathbf{wK4}$, replacing $4$ with:

$$\Box \varphi \land \varphi \rightarrow \Box \Box \varphi .$$

Esakia also showed $\mathbf{K4}$ axiomatizes the $T_d$-spaces.
The most important result from Esakia [1981] is completeness of $\mathbf{GL}$ w.r.t. the \textit{scattered spaces}:

Every subspace has an isolated point.

Löb’s axiom: $\Diamond \varphi \rightarrow \Diamond (\varphi \land \neg \Diamond \varphi)$. Topologically speaking:

$$d(X) = d(X \setminus d(X)).$$

A canonical example of a scattered space is any ordinal with the interval topology (sub-basis of open rays).

\textbf{Theorem (Abashidze 1985; Blass 1990)}

$\mathbf{GL}$ is complete with respect to $\omega^\omega$ with interval topology.
- \( \text{GLP}^0 \) is also complete w.r.t. a polytopological space defined on \( \epsilon_0 \).

- Infinitely many finer and finer topologies on \( \epsilon_0 \):
  - \( \tau_0 \) is the ‘left topology’;
  - \( \tau_1 \) is the interval topology (a.k.a. ‘order topology’);
  - \( \tau_2 \) adds to \( \tau_1 \) subbasic open sets \( \{ \alpha : l(\alpha) > \beta \} \)

- In general, \( \tau_{n+1} \) adds to \( \tau_n \) subbasic sets

\[ \{ \alpha : l^n(\alpha) > \beta \}. \]

**Theorem (Icard 2011)**

\( \text{GLP}^0 \) is complete with respect to \( (\epsilon_0, \{\tau_n\}_{n<\omega}) \).
In very recent work, Fernández-Duque and Joosten [2013] generalize the previous theorem to arbitrary ordinals.

$L^0_\omega$ is a special case of $L^0_\Lambda$, and the ‘logarithm’ function $l$ must be able to iterate into the transfinite.

They generalize Ignatiev’s ‘universal frame’ for $L^0_\Lambda$, and by eliminating ‘non-root’ points and generalizing the topologies in the previous slide, they obtain topological completeness.

Theorem (Fernández-Duque and Joosten 2013)

$\text{GLP}^0_\Lambda$ is complete w.r.t. frame and topological semantics.
Definition
A polytopological space \((X, \{\tau_n\}_{n<\omega})\) is a \textbf{GLP-space} if, for all \(n\):

- \(\tau_n\) is scattered;
- \(\tau_n \subseteq \tau_{n+1}\);
- \(d_n(X) \in \tau_{n+1}\).

Are there non-trivial GLP-spaces?
Beklemishev et al. [2010] proved topological completeness of \textbf{GLB} w.r.t. \textbf{GLP}-spaces, using relational semantics established in Beklemishev [2010].

$\tau_0$ is the upset topology (standard \textbf{GL} model).

$\tau_1$ is the interval topology.
GLP-spaces

Ordinal Models

Fact (Beklemishev et al. 2010)

If \((X, \tau_0, \tau_1)\) is a GLB-space and \(\tau_0\) is first-countable and Hausdorff, then \(\tau_1\) must be discrete.

Fact (Beklemishev 2011)

If \(\Omega \geq \aleph_\omega\) and \(V = L\), then GLB is complete w.r.t. \((\Omega, \tau_0, \tau_1)\), where \(\tau_0\) is interval topology and \(\tau_1\) is club topology.

- If \(cf(\alpha) \leq \aleph_0\), then \(\alpha\) is isolated;
- If \(cf(\alpha) > \aleph_0\), then any neighborhood of \(\alpha\) contains a club.
- Blass [1990] showed incompleteness of GL w.r.t. \(\tau_1\) is equiconsistent with existence of a weakly Mahlo cardinal.
Even more recently Beklemishev and Gabelaia [2013] have settled a number of questions.

They note it is independent of ZFC whether $\tau_2$ can be non-trivial on an ordinal space, but show there are assumptions that guarantee all $\tau_n$ are non-discrete.

The question of whether there are set-theoretic assumptions that would give ordinal completeness of GLP is open.

Beklemishev and Gabelaia [2013] do show topological completeness of GLP w.r.t. general GLP-spaces, but (in their own words) “it is not an example of a natural GLP-space.”
The study of topological models of provability logic combines work in proof theory, topology, algebra, set theory, and modal logic, exposing some mathematically interesting structures.

It also promises a better grip on the polymodal provability logic GLP, providing a simple non-arithmetical interpretation.

Current work in Moscow and Barcelona is focused on using these models to extend Beklemishev’s original ordinal analysis of PA to stronger theories and larger ordinals.
Thanks for your attention!
References


