Logic Seminar, Fall 2015 (Phil 391 or Math 391)

Introduction to the EFI workshop, Part II

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Abstract: Following a brief review, this is a continuation of the material presented on Sept. 22. The Notes for that are posted on the Logic at Stanford web page http://www-logic.stanford.edu [*note the dash*] under "Seminar". For the EFI workshop series, see http://logic.harvard.edu/efi.php.

Topics for Part II of this introduction include constructible and relativized constructible sets, model theory for set theory, the (Los) Ultrapower Theorem, some "small" large cardinals, measurable cardinals, and Scott's theorem that the existence of the latter implies ($V \neq L$).

Books on Reserve in Tanner Library:

K. Hrbacek, Introduction to Set Theory (3d edn.)

K. Kunen, Set Theory. An introduction to independence proofs

J. R. Shoenfield, Mathematical Logic

A. Kanamori, The Higher Infinite (2nd. edn.)

K. Gödel, Collected Works Vol. II. Publications 1938-1974.

G. H. Moore, Zermelo's Axiom of Choice

1. Transfinite recursion. V = the class of all sets, ORD = the class of all ordinals (aka ON or On). Classes and functions on classes are given by defining formulas on V.

Trans(x) ("x is transitive") $\Leftrightarrow \bigcup x \subseteq x \Leftrightarrow x \subseteq \wp(x)$. The *transitive closure* tc(x) = least transitive y with $x \subseteq y$.

Transfinite recursion on V w.r.t. $\in$: $(\forall G{:}V^2 \to V)(\exists! F{:}V \to V)\forall x[F(x) = G(x, F|tc(x))]$.

ORD is well-ordered by $\in$; on ORD write $<$ for $\in$. $\alpha, \beta, \ldots$ range over ORD. $\forall\alpha[\alpha = \{\beta : \beta < \alpha\}]$. $\alpha + 1 = sc(\alpha) = \alpha \cup \{\alpha\}$. If $a \subset$ ORD, $\bigcup a = \sup(a)$; if a is unbounded, sup(a) is a limit ordinal.

Recursion on ORD: $(\forall G{:}ORD \times V \to V)(\exists! F{:}\ ORD \to V]\forall\alpha[F(\alpha) = G(\alpha, F|\alpha)]$.Typical recursions on ORD to V: $(\forall G, H, c)$ $(\exists! F{:}\ ORD \to V)[F(0) = c \wedge F(\alpha{+}1) = G(\alpha, F(\alpha))$ (all α) $\wedge\ F(\alpha) = H(\alpha, F|\alpha)$ for limit α].

2. The Cumulative Hierarchy: $V_0 = 0$, $V_{\alpha+1} = \wp(V_\alpha)$ and $V_\alpha = \bigcup V_\beta\ [\beta < \alpha]$ for limit α. Each V_α is transitive and $\beta \leq \alpha$ implies $V_\beta \subseteq V_\alpha$.

Theorem. $V = \bigcup_\alpha V_\alpha$.

This is the "intended model" of ZF and satisfies AC.

For each set x, *rank*(x) $=_{def}$ the unique α with $x \in V_{\alpha+1} - V_\alpha$.

3. Standard models. Models are sets or classes M with $\in$ restricted to M. (More generally for the language of set theory, (M, R) with R binary on M.)

Given $\varphi(\underline{x})$ with free variables $\underline{x} = x_1,\dots,x_n$ and $\underline{a} = a_1,\dots,a_n \in M$, write $M \models \varphi[\underline{a}]$ when the model M satisfies φ at the assignment of $\underline{a}$ to $\underline{x}$. Write $\varphi^M(\underline{a})$ for the *relativization* of all quantifiers in φ to M, i.e. replace $(\exists y)(\dots)$ in φ by $(\exists y \in M)(\dots)$, i.e. $(\exists y)[y \in M \wedge \dots]$ and replace each $(\forall y)(\dots)$ in φ by $(\forall y \in M)(\dots)$, i.e. $(\forall y)[y \in M \rightarrow \dots]$. When the a_i are in M, this is equivalent to the satisfaction relation.

The satisfaction relation is definable in ZF for arbitrary *sets* M, formulas φ and assignments $\underline{a}$ to its free variables. If M is a class the satisfaction relation is definable in ZF only for each given formula and its subformulas.

If M is a set, Def(M) (or $\wp^{Def}(M)$) = the set of all subsets of M definable from parameters in M, i.e. it consists of all sets of the form $\{x \in M : \varphi^M(x, \underline{a})\}$ where the a_i are all in M.

4. The Constructible Hierarchy: $L_0 = 0$, $L_{\alpha+1} = Def(L_\alpha)$ and $L_\alpha = \bigcup L_\beta\ [\beta < \alpha]$ for limit α. Each L_α is transitive and $\beta \le \alpha$ implies $L_\beta \subseteq L_\alpha$. Let $L = \bigcup_\alpha L_\alpha$; a set is called *constructible* if it belongs to L. The *Axiom of Constructibility* is (V = L).

Theorem. (i) L is a model of ZF + (V = L). (ii) ZF + (V = L) proves AC and GCH.

5. Relativized constructibility. Given a set u, define $L_0(c) = tc(\{u\})$ and then $L_\alpha(u)$ for arbitrary α as for the L_α's. L(u) is their union. In ZF, L(u) need not satisfy AC or CH, depending on u. Of particular interest in the literature of large cardinals is $L(\mathbb{R})$, i.e. essentially $L(2^{\mathbb{N}})$. $L(\mathbb{R})$ includes all sets dealt with in *Descriptive Set Theory* and more.

NB. There is also a notion of L[u] in the literature that is different from L(u); this will never be used here.

6. Elementary equivalence and extension. M_1 and M_2, are *elementarily equivalent*, and write $M_1 \equiv M_2$, iff they satisfy the same sentences, i.e. for each φ, $M_1 \models \varphi$ iff $M_2 \models \varphi$. M_2 is an *elementary extension* of M_1, and write $M_1 \prec M_2$, if for each $\varphi(\underline{x})$ and each $\underline{a}$ in M_1 we have $M_1 \models \varphi[\underline{a}] \Leftrightarrow M_2 \models \varphi[\underline{a}]$.

More generally, $j: M_1 \prec M_2 \Leftrightarrow j: M_1 \rightarrow M_2$ is an injection and for each $\underline{a}$ in M_1 we have $M_1 \models \varphi[\underline{a}]$ iff $M_2 \models \varphi[j(\underline{a})]$.

7. Convention. Unless otherwise noted, from now on we work in ZFC.

8. Löwenheim-Skolem-Tarski theorems. The Löwenheim-Skolem theorem tells us that if a first order theory has an infinite model then it has a countable model. Tarski sharpened this both *downwards* and *upwards* in terms of the $\prec$ relation.

Lemma. Suppose $M_1 \subseteq M_2$ and for each $\varphi(\underline{x}, y)$, and each $\underline{a}$ in M_1 there exists b in M_2 such that $M_2 \models \varphi[\underline{a}, b]$ iff there exists b in M_1 such that $M_2 \models \varphi[\underline{a}, b]$. Then $M_1 \prec M_2$.

f_φ is a *Skolem function* for M_2 and $\varphi(\underline{x}, y)$ if $M_2 \models \exists y\varphi[\underline{a}, y] \Rightarrow M_2 \models \varphi[\underline{a}, f_\varphi(\underline{a}))]$.

Theorem. (Downwards LST) Suppose $\aleph_0 \leq \kappa < \mathrm{Card}(M_2)$, and that $A \subseteq M_2$ with Card(A) $= \kappa$. Then there exists $M_1 \prec M_2$ with $A \subseteq M_1$ and $\mathrm{Card}(M_1) = \kappa$.

This is proved by closing under all the Skolem functions. (The upwards LST uses a suitable form of compactness instead.)

Montague-Vaught Theorem. There are arbitrarily large α with $V_\alpha \prec V$.

A weaker form is the *Lèvy Reflection Principle.* That is the beginning of the idea that no property φ (even higher order) can capture V.

9. Boolean algebras (B. A.) are all $(B, \wedge, \vee, -, \leq, 0, 1)$ satisfying commutative, associative and distributive laws in which $x \leq y \Leftrightarrow x \wedge y = x \Leftrightarrow x \vee y = y$, and for all x, $0 \leq x \leq 1$, $x \wedge (-x) = 0$, and $x \vee (-x) = 1$.

The *complete* B. A. on a non-empty set I is $(\wp(I), \cap, \cup, -, \subseteq, 0, I)$, in which $-$ is complement relative to I. *Stone Representation Theorem*: Every B. A. is isomorphically embeddable in a complete B. A.

10. Filters. For a Boolean algebra B, a non-empty subset F of B is called a *filter* if x, y in F implies $x \wedge y$ in F, and x in F and $x \leq y$ implies y in F. (*Ideals* are the dual notion.)

By a *filter (ultrafilter) F on I* is meant a filter (ultrafilter) in $\wp(I)$; F is called *principal* if for some $i \in I$, $F = \{X \subseteq I : i \in X\}$, otherwise *non-principal.*

Given a filter F on B we can define a B.A. B/F and an onto homomorphism $h: B \rightarrow B/F$. If B and B' are B.A.s and $h: B \rightarrow B'$ is an onto-homomorphism then $F = \{x \in B: h(x) = 1\}$ is a filter on B and $B' \cong B/F$.

By an *ultrafilter* is meant a maximal filter F; a filter on B is an ultrafilter iff $B/F \cong \{0, 1\}$. We use the letter 'U' for ultrafilters.

11. Ultrapowers. Let M be a model and I a non-empty index set; $f \in M^I \Leftrightarrow f: I \rightarrow M$. Given an ultrafilter U on I, define $(f \sim_U g)$ iff $\{i \in I; f(i) = g(i)\} \in U$; this is an equivalence relation on M^I, with equivalence classes f/U. Define $f/U \in_U g/U$ iff $\{i \in I: f(i) \in g(i)\} \in U$. By M^I/U is meant the structure of equivalence classes with the $\in_U$ relation. (There is a natural generalization of this to *ultraproducts.*)

Los' Theorem. Suppose U is an ultrafilter on I. Given any $\varphi(x_1,\ldots,x_n)$ and $f_1,\ldots,f_n$ in M^I we have $M^I/U \vDash \varphi[f_1/U,\ldots,f_n/U] \Leftrightarrow \{i \in I : M \vDash \varphi[f_1(i),\ldots,f_n(i)]\} \in U$. Let j be the *canonical embedding* of M into M^I/U, i.e. let $j(a) = a^*/U$ where $a^*(i) = a$. Then $j: M \prec M^I/U$.

12. Cofinality; regular and singular cardinals. κ^+ is the least cardinal greater than κ. A *successor cardinal* is one of the form κ^+, otherwise it is a *limit cardinal.* The *cofinality* of a limit ordinal α, $cf(\alpha)$, is the least β such that α is the limit of a β-termed strictly increasing sequence of ordinals. Then $cf(\alpha)$ is a cardinal and $cf(cf(\alpha)) = cf(\alpha)$. A cardinal κ is *regular* if $cf(\kappa) = \kappa$, otherwise it is *singular*, i.e. $cf(\kappa) < \kappa$. Each successor cardinal is regular, so if κ is singular it must be a limit cardinal. *But there may exist limit cardinals that are regular.* Ex: $cf(\aleph_0) = \aleph_0$, $cf(\aleph_\omega) = \omega = \aleph_0$.

13. Inaccessible and Mahlo cardinals. A cardinal κ is called *weakly inaccessible* if it is a regular limit cardinal; for such κ, L_κ is a model of ZFC. So, by Gödel's second incompleteness theorem, we can't prove the existence of weakly inaccessible cardinals in ZFC if it is consistent.

A weakly inaccessible κ is called *strongly inaccessible* if, also, for every cardinal $\lambda < \kappa$ we have $2^\lambda < \kappa$. *From now on, "inaccessible" is meant in the strong sense*. If κ is an inaccessible cardinal then V_κ is a model of ZFC.

Suppose there are arbitrarily large inaccessible cardinals. Then we could enumerate these in a transfinite sequence ι_α indexed by $\alpha \in ORD$; by a simple argument we can get arbitrarily large fixed points $\iota_\alpha = \alpha$. In the early 1900s, Mahlo considered the possible existence of arbitrarily large *regular* fixed points. Then we could enumerate those, and so on. This leads to the idea of a hierarchy of α-Mahlo cardinals.

14. "Small" large cardinals. A "large" cardinal κ is said to be "small" if it is given by a property $P(\kappa)$ such that if it holds in V then it holds in L. The properties of being inaccessible, 1-Mahlo, 2-Mahlo, …, α-Mahlo, are all properties of small large cardinals. Other notions that lead only to small large cardinals are*: indescribable cardinal* and *weakly compact cardinal*.

15. Measurable cardinals and "large" large cardinals. An ultrafilter U on κ is called *<κ-complete* if U is closed under all intersection of collections of subsets of κ of cardinality $< \kappa$. κ is *measurable* if there exists a non-principal $< \kappa$ complete ultrafilter U on κ. In that case we can define a "measure" $\mu: \wp(\kappa) \to \{0, 1\}$ such that $\mu(X) = 1$ iff $X \in U$, so that (i) for each $\alpha < \kappa$, $\mu(\{\alpha\}) = 0$, (ii) $\mu(\kappa) = 1$, and (iii) μ is *<κ- additive*, i.e. for any collection $X_i[i < \lambda]$ of pairwise disjoint subsets of κ for $\lambda < \kappa$, $\mu(\bigcup X_i) = \sum \mu(X_i)$.

Scott's Theorem. If $(\exists\kappa)(\kappa$ is measurable) then $V \neq L$.

Proof by contradiction using the canonical injection $j: V \prec V^\kappa/U$. (This looks a bit weird, because $M = V^\kappa/U$ is a subclass of V, and we have non-trivial $j: V \prec M$.)

Measurable cardinals are the first example of "large" large cardinals, if they exist at all.