Logic Seminar, Fall 2015 (Phil 391 or Math 391)

Seminar times, location: Tuesdays 4:30-5:50, Room 380-381T.

Seminar leader: Solomon Feferman; Email: Feferman@stanford.edu, Office: 380-382G, by appointment.

Enrollment: Students enroll in Phil 391 or Math 391, and may sign up for credit of 1-3 units, either for a letter or CR grade. In order to obtain a letter grade, the student must offer either at least one talk in the seminar or a paper on material relevant to the seminar; neither need be original material. The instructor should be consulted as to the choice. In order to obtain a CR grade, the student should attend at least 75% of the seminar meetings. In either case, participation in discussions is strongly encouraged. (See Course Works for further information.)

Seminar plans: This quarter and next we shall largely be making use of materials from the Exploring the Frontiers of Incompleteness (EFI) workshops held at Harvard in 2011-2012. The EFI site (http://logic.harvard.edu/efi.php#multimedia) provides background materials, a series of eleven individual lectures and/or individual workshop materials. Some of the topics discussed are the incompleteness phenomena for set theory, the programs for new axioms—in particular large cardinal axioms—to overcome incompleteness, the axiom of determinacy, and the continuum problem. This material is of great active logical and philosophical interest. It will be assumed that participants have a background in first order logic through the completeness theorem as well as introductory set theory. (Useful texts will be put on reserve in Tanner Library; there is also much online introductory material.)

In the first and second seminar meetings (Sept. 22 and 29) I will give some historical background followed by an informal review of axiomatic set theory, the cumulative hierarchy, the constructible hierarchy, inaccessibles, Mahlo cardinals and other "small" large cardinals, and the first "large" large cardinals (measurables). On Oct. 6, Rick Sommer will describe the method of forcing and its applications to independence results. That will be followed by two lectures by J. T. Chipman on Descriptive Set Theory and the Axiom of Determinacy.

Review material. See articles in the Stanford Encyclopedia of Philosophy (SEP), beginning with Joan Bagaria, Set Theory, http://plato.stanford.edu/entries/set-theory/, especially through sec. 5 plus the Appendix.

Informal ("Naïve") set theory. Cantor 1870s and on. Sets, membership ($x \in A$), sets as independently existing entities ("platonism"), extensionality. Ordered pairs, n-tuples. Relations and functions as sets, partial orderings, linear orderings. $A \cup B$, $A \times B$, A^B, $\wp(A)$, $2 = \{0, 1\}$.

Cardinal equivalence $A \sim B$, $A \preccurlyeq B$, $A \prec B$. Cardinal numbers ($\kappa, \lambda, \ldots$) as equivalence types of $\sim$; Card(A). $\kappa < \lambda$, $\kappa + \lambda$, $\kappa \times \lambda$, κ^λ. $+$ and $\times$ are commutative and associative.

Well-founded relations, well-orderings. Proof by induction and definition by recursion on well-founded relations. Any two well-orderings are comparable. Ordinal numbers (α, β,…) as isomorphism types of well-orderings. $\alpha < \beta$, $\alpha + \beta$, $\alpha\beta$, α^β. The ordinal ω. Successor ordinals, limit ordinals. + and $\times$ are associative but not commutative, for example $1 + \omega = \omega$ but $\omega < \omega + 1$.

The transfinite. Finite cardinals, infinite cardinals, $\aleph_0 = \mathrm{Card}(\mathbb{N})$, $\mathbf{c}$ = the cardinal of the continuum, i.e. $\mathrm{Card}(\mathbb{R}) = \mathrm{Card}(2^{\mathbb{N}}) = \mathrm{Card}(\wp(\mathbb{N})) = 2^{\aleph_0}$. $\aleph_0 < \mathbf{c}$; in general, $\kappa < 2^\kappa$.

Cantor's Continuum Hypothesis (CH): There is no κ with $\aleph_0 < \kappa < \mathbf{c}$, i.e. every infinite subset X of $\mathbb{R}$ is either ~ N or ~ R. (Originally claimed to be a theorem.)

Cantor's Well-Ordering Hypothesis (WO): Every set can be well-ordered. (Originally stated as a law of thought). This implies that the cardinal numbers are linearly ordered by < ("trichotomy" law). Also for every infinite κ, λ, we have $\kappa + \lambda = \kappa \times \lambda = \max(\kappa, \lambda)$; in particular, $\kappa + \kappa = \kappa \times \kappa = \kappa$.

The scale of transfinite cardinals: For $\alpha > 0$, $\aleph_\alpha$ = the least cardinal greater than $\aleph_\beta$ for all $\beta < \alpha$. (Assumes WO.) So we have the second form of CH: $2^{\aleph_0} = \aleph_1$. The Generalized Continuum Hypothesis (GCH), $2^{\aleph_\alpha} = \aleph_{\alpha+1}$.

Zermelo's Axiom of Choice (AC) (1904)**:** For each set A there is a function F: $\wp(A) \to A$ such that for each non-empty $X \subseteq A$, we have $F(X) \in X$. This is equivalent to the statement that every set A of non-empty disjoint sets has a choice set. Also AC is equivalent to WO. In 1908, Zermelo introduced his axiomatic system of set theory to respond to critics of the 1904 publication. His axioms also showed how the paradoxes of naïve set theory (Russell, Cantor, Burali-Forti) could apparently be avoided. Zermelo's system allowed urelements. The Aussonderungs (Separation) axiom of his system uses the informal idea of a definite property. Also, the Zermelo system can't prove the existence of the limit of the sequence $F(\alpha)$, where $F(0) = \aleph_0$ and $F(n+1) = 2^{F(n)}$.

The Zermelo-Fraenkel axiom system ZF. The language has one sort of variable a, b, …,x, y, z ranging over arbitrary sets or the "universe of all sets". There are two basic relation symbols, = and $\in$. The underlying logic is that of classical first-order logic with identity. The axioms are: 1. Extensionality; 2. Empty set, 0; 3. Unordered pair, $\{x, y\}$; 4. Power set, $\wp(a)$; 5. Union set, $\bigcup a$, or $\bigcup x\ [x \in a]$, 6. Infinity, ω; 7. Separation (axiom scheme), $\{x \in a : \varphi(x)\}$, φ with possible parameters; 8. Replacement: for each first-order definable function given as F(x) = the unique y such that $\varphi(x, y)$ with domain $x \in a$, the range of F = $\{y : (\exists x)[x \in a \wedge \varphi(x, y)]\}$ exists as a set; 9. Foundation, every non-empty set contains an element that is minimal in the $\in$ relation, i.e. the universe is well-founded with respect to the membership relation. (Note: Replacement implies Separation.) We take ZFC = ZF + AC.

Some consequences of ZF. Ordered pairs; define $(x, y) = (\{x\}, \{x, y\})$. Then relations and functions with bounded domains and ranges can be identified with sets. Ordered n-

tuples. A set x is called *transitive* if members of members are members, i.e. the union of x equals x. The *transitive closure* of x is the least transitive set containing x.

Ordinals are defined in such a way that each ordinal is the set of all its predecessors. Ordinals are transitive and linearly ordered by $\in$, which is then the same as $<$ on ordinals. $\alpha \leq \beta$ iff $\alpha \subseteq \beta$. The successor of an ordinal α, $sc(\alpha) = \alpha \cup \{\alpha\}$. If a is any set of ordinals then $\bigcup a$ or *sup*(a) is the least ordinal greater than or equal to each $\alpha \in a$; if a is unbounded, this is called the *limit of a.* Every well-ordered set is isomorphic to an ordinal under $<$. The set of natural numbers is identified with $\omega = \{0, 1, 2,\ldots\}$.

Cardinals are defined as ordinals that are not ~ to any smaller ordinal. Real numbers are defined using Dedekind or Cantor's approach. It is often claimed that ZFC is a foundation for "all" of mathematics, but there are exceptions.

Classes. In ZF, classes are treated as extensions of properties, $A = \{x : \varphi(x)\}$; a class is *proper* if its extension is not a set. In particular, we have the proper classes $V = \{x : x = x\}$ and $ORD = \{x : x \text{ is an ordinal}\}$. As an alternative, we can take a language with both set variables and class variables as basic. The theory of sets and classes of von Neumann, Bernays and Gödel, NBG (or GB) is a conservative extension of ZF, and the same with AC added. (A system MK due to Morse and Kelley, is much stronger.)

Consequences of Foundation. A class A is called *progressive* if $\forall x[x \subseteq A \rightarrow x \in A]$. The Foundation Axiom implies that *if A is progressive then A = V*. This is a form of proof by induction on the universe. Its restriction to ORD gives us proof by induction on the ordinals. Transfinite induction on the universe can be used to define $F: V \rightarrow V$ by $F(x) = G(\{F(y): y \in x\})$ (transfinite recursion). Similarly for definition by transfinite recursion on the ordinals, $F: ORD \rightarrow V$. Usually, for the latter, F(0) is defined outright, $F(sc(\alpha)) = G(F(\alpha))$, and for α a limit ordinal, $F(\alpha) = H(\{F(\beta) : \beta < \alpha\}$. For example, the ordinal operations may be given by transfinite recursion: $\gamma + 0 = \gamma$, $\gamma + sc(\alpha) = sc(\gamma + \alpha)$, and for limit α, $\gamma + \alpha = \sup\{\gamma + \xi : \xi < \alpha\}$, and similarly for + and exponentiation.

The Cumulative Hierarchy is defined by $F(0) = 0$, $F(\alpha + 1) = F(\alpha) \cup \wp(F(\alpha))$, and for limit α, $F(\alpha)$ is the union of all $F(\beta)$ for $\beta < \alpha$. We write V_α for $F(\alpha)$. It is then proved that V = the union of all V_α for $\alpha \in ORD$. Informally speaking, *the Cumulative Hierarchy is a model of ZF* (Zermelo 1930).

Standard Models. Consider structures $M = (A, \in|A)$ where A is a set or class. A subset X of A is definable from parameters in A if $X = \{x \in A : M \vDash \varphi(x, y_1,\ldots,y_n)\}$ for some formula φ and elements $y_1, \ldots, y_n$ of A. If A is a set, Def(A) is the set of all definable subsets of A.

The Constructible Hierarchy. The sequence of sets L_α is defined by recursion by $L_0 = 0$, $L_{\alpha+1} = L_\alpha \cup Def(L_\alpha)$ and for limit α, L_α is the union of all L_β for $\beta < \alpha$. L is defined as the union of all L_α, and a set is called *constructible* if it belongs to L. Gödel (1938) proved that $(L, \in)$ is a model of ZF + (V = L), and V= L implies AC + GCH.