Introduction to Classical Descriptive Set Theory

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"The efforts that I exerted led me to the following totally unexpected discovery: there exists a family [...] consisting of effective [i.e., definable] sets, such that one does not know and one will never know whether every set from this family, if uncountable, has the cardinality of the continuum, nor whether it [has the property of Baire], nor whether it is measurable. [...] This is the family of the projective sets of Mr. H. Lebesgue. It remains but to recognize the nature of this new development." (Luzin (1925))

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Kanamori summarizes the subject matter of Descriptive Set Theory succinctly:

> *"Descriptive Set Theory is the definability theory of the continuum."*

Concerning the origins of the theory, Moschovakis writes in the opening lines of his textbook:

> *"The roots of Descriptive Set Theory go back to the work of Borel, Baire, and Lebesgue around the turn of the 20th century, when the young French analysts were trying to come to grips with the abstract notion of a function. "*

In this section, we provide a brisk history of the relevant results of Borel, Baire, Lebesgue, and, last, the Russians Luzin and Suslin. There are three goals of this presentation:

- To provide a brief historical and technical introduction to the study of definable sets of reals and their regularity properties
- To make explicit the relevant *mathematical* motivations of this theory (as opposed to *logical* or *metamathematical* motivations), and
- To motivate the relationship between definability and principles of definable determinacy

Émile Borel (1871-1956)

Borel's 1898 book, *Leçons sur la théorie des fonctions* ("Lessons on the theory of functions"), introduced the contemporary notion of a measure on the real line:

- The development of *analytical geometry* motivated the need to define a general notion of measure for (subsets of) $\mathbb{R}^n$
- One of Borel's key new ideas was employing *countable additivity* and using this to define the Borel sets
- Beginning with a family of sets of an obvious measure, such as open or closed intervals on $[0, 1]$, and a function assigning to each interval its length, he recursively expands the function's domain of definition in stages to sets whose complements are previously defined and sets that are the unions of previously defined disjoint sets

René-Louis Baire (1874-1932)

Baire's 1899 thesis, *Sur le fonctions de variables réelles* ("On the functions of real variables"), analyzed real functions in terms of a proper transfinite hierarchy—the first such analysis since Cantor:

- *Baire class* 0 consists of the continuous real functions
- For countable ordinals $\alpha > 0$, *Baire class* α consists of functions f not in any previous class but are nonetheless pointwise limits of sequences of functions $g_0, g_1, g_2, \ldots$ from previous classes: i.e., $f(x) = \lim_{n\to\infty} g_n(x)$.
- Further, he proved two facts concerning the functions in these classes (i.e., the *Baire functions*):
  - They are closed under pointwise limits, and
  - There exist non-Baire real functions

Henri Lebesgue (1875-1941)

Lebesgue's 1902 thesis, *Intégral, longueur, aire* ("Integral, length, area"), is the source of modern measure theory. His concept of measurable set subsumed the Borel sets and his analytic definition of measurable function subsumed the Baire functions.

Further elaboration of these ideas in his 1905 *Sur les fonctions représentables analytiquement* ("On the analytically representable functions") yielded two major results (stated on the following page) of enduring interest. These results were enabled by establishing a correlation between the Borel sets and the Baire functions: the former are exactly the pre-images $\{x|f(x) \in O\}$ of open intervals O by Baire functions f.

Here are the theorems:

- For every countable ordinal α, there is a Baire function of class α. I.e., the Baire (and hence Borel) hierarchy is proper.
- There is a Lebesgue measurable function that is not in any Baire class. Hence there are non-Borel Lebesgue measurable sets.

Between 1905 and 1915, the work of the French analysts also resulted in a codification of the *regularity properties*—properties indicative of "well-behaved" sets of reals. It would be, however, a Russian student in Paris, Nikolai Luzin, whose initiative led to the subject of descriptive set theory as a distinct discipline.

Nikolai Luzin (1883-1950) & Mikhail Suslin (1894-1919)

While investigating the results of Lebesgue described above, one of Luzin's students, Suslin, made an important discovery concerning the *projections* of Borel sets. I.e., for $Y \subseteq \mathbb{R}^{k+1}$, the projection of Y is:

$$p[Y] = \{< x_1, ..., x_k > \in \mathbb{R}^k | \exists y (< x_1, ..., x_k, y > \in Y)\}$$

Suslin discovered a mistaken assumption of Lebesgue's to the effect that projections of (G_δ) Borel subsets of the plane are also Borel by constructing a counterexample. Hence the Borel sets are *not* closed under projection (it turns out that, more generally, they are not closed under continuous images).

Suslin's 1917 *Sur une définition des ensembles mesurables B[orel] sans nombres transfinis* ("On a definition of Borel measurable sets without transfinite numbers") uses something he calls "operation $\mathcal{A}$" to directly define the sets used in his counterexample—the analytic sets. We will, however, use the more modern definition of them:

$X \subseteq \mathbb{R}$ is *analytic* if there is a closed $Y \subseteq \mathbb{R} \times \omega^\omega$ such that, for all x, $x \in X \leftrightarrow \exists y \in \omega^\omega \langle x, y \rangle \in Y$.

In effect, operation $\mathcal{A}$ had allowed Suslin to "factor out" and collect the first coordinates of the ordered pairs in countable intersections of open sets in two dimensions by means of uncountable unions.

Suslin's work on analytic sets produced four important theorems:

- Every Borel set is analytic
- There is an analytic set that is not Borel
- A set of reals is Borel iff both it and its complement are analytic
- Every analytic set is Lebesgue measurable, has the property of Baire, and has the perfect set property

Note the analogy here, for the first three, to later results. I.e., when "Borel" is replaced by "recursive" and "analytic" is replaced by "recursively enumerable." We will return to this point later.

In the rest of the presentation, we use the preceding historical survey to motivate central concerns of contemporary descriptive set theory:

- The hierarchy of definable sets of reals: Borel, Projective, and those in $L(\mathbb{R})$
- Regularity properties of definable sets of reals: Lebesgue measurability, Baire property, and the perfect set property
  - Note that I omit the "structural properties," such as Uniformization, for the sake of time
- The limitative results in the field and motivating principles of definable determinacy

It is convenient to use the set of functions from ω to ω—ω^ω (the "Baire space")—instead of $\mathbb{R}$. This is for a few topological reasons:

- Both ω^ω and $\mathbb{R}$ are *perfect Polish spaces* (no isolated points, completely metrizable, and contain a countable dense subset):
  - The *Baire Category Theorem* (by the second listed property) ensures that every non-empty open set in either space is non-meagre (i.e., non-negligible in a sense we will make precise)
  - $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$ and $\mathbb{R}\backslash\mathbb{Q}$, the irrationals, form an uncountable subset
  - Under the subspace topology inherited from $\mathbb{R}$, the irrationals are topologically equivalent to ω^ω

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- So we picture ω^ω as the set of all sequences of non-negative integers
- The basic open sets are formed from a countable subset $\omega^{<\omega}$—the set of all finite strings of positive integers:
  - For $s \in \omega^{<\omega}$, the set of sequences extending s form the open set N_s
  - The set of such N_s generate the product topology on ω^ω
- Last, since each of ω^ω and $\mathbb{R}$ are perfect Polish spaces, so are n-dimensional versions for $n < \omega$

We define the Borel sets $\mathcal{B}$ as the smallest set of ω^ω with the following properties:

- Every open set is in $\mathcal{B}$
- If $B \in \mathcal{B}$, then $(\omega^\omega \backslash B) \in \mathcal{B}$
- If $B_i \in \mathcal{B}$ for all $i \in \omega$, then $\bigcup_i B_i \in \mathcal{B}$

I.e., the Borel sets are recursively generated from the open sets by the operations of complementation and countable union—hence, they form a σ-algebra.

We define the *Borel hierarchy* inductively as follows. Suppose $A \subseteq \omega^\omega$ and α an ordinal with $\alpha < \aleph_1$, then:

- A is $\mathbf{\Sigma}^0_1$ iff A is open
- A is $\mathbf{\Pi}^0_1$ iff $\omega^\omega \backslash A$ is $\mathbf{\Sigma}^0_1$ (i.e., A is closed)
- For $\alpha > 1$, A is $\mathbf{\Sigma}^0_\alpha$ iff, for each $A_n \in \mathbf{\Pi}^0_\beta$ with $\beta < \alpha$, $A = \bigcup_{n \in \omega} A_n$
- For $\alpha > 1$, A is $\mathbf{\Pi}^0_\alpha$ iff $\omega^\omega \backslash A$ is $\mathbf{\Sigma}^0_\alpha$
- A is $\mathbf{\Delta}^0_\alpha$ iff A is both $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$

Hence, a set $B \in \mathcal{B}$ just in case, for $\alpha < \aleph_1$, B is either $\mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$—further, the least such α is the *Borel rank* of B. Recall that Lebesgue's 1905 showed that the Borel hierarchy is proper (i.e., new sets appear at each level) via identification of each class of the hierarchy with a class of Baire functions. The classical naming conventions, beginning with F_σ and G_δ, correspond respectively to $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_2$.

Recall now Suslin's discovery that there are (lower dimensional) projections of Borel subsets of the plane that are not themselves Borel and hence that $\mathcal{B}$ is not closed under projection. Further, that he introduced the study of the *analytic* sets by consideration of the projections of closed $\mathbf{\Pi}^0_2$ subsets of the plane along an axis.

We now use this operation to define a hierarchy in a manner analogous to the use of countable union in defining the Borel hierarchy.

We define the *Projective hierarchy* inductively as follows. Suppose $A \subseteq \omega^\omega$ and $n \in \omega$, then:

- A is $\mathbf{\Sigma}^1_0$ iff A is open
- A is $\mathbf{\Pi}^1_0$ iff A is closed
- A is $\mathbf{\Pi}^1_n$ iff $\omega^\omega \backslash A$ is $\mathbf{\Sigma}^1_n$
- A is $\mathbf{\Sigma}^1_{n+1}$ iff $A = p[A']$ for A' in $\mathbf{\Pi}^1_n$
- A is $\mathbf{\Delta}^1_n$ iff A is both $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$

To define the analytic sets that emerged from Suslin's discovery, it turns out that taking A to be closed is sufficient to ensure $p[A] \notin \mathcal{B}$. In our hierarchy, $\mathbf{\Sigma}^1_1$ sets are the projections of closed subsets of ω^ω—accordingly, for $A \subseteq \omega^\omega$, A is analytic just in case A is $\mathbf{\Sigma}^1_1$.

Also recall Suslin's claim that, for $A \subseteq \omega^\omega$, A is Borel iff both A and $\omega^\omega \backslash A$ are analytic (i.e., A is both analytic and co-analytic). This can now be expressed as the claim that the sets in $\mathcal{B}$ are exactly the $\mathbf{\Delta}^1_1$ sets.

We have so far characterized the Borel and Projective hierarchies in topological terms rather than in terms of definability. Nonetheless, the projective sets, for instance, are exactly those that are definable with real parameters in second-order arithmetic:

- The superscript in '$\Box^1_n$' (where $\Box \in \{\mathbf{\Sigma}, \mathbf{\Pi}\}$) denotes quantification over real numbers
- For sets in classes denoted with '$\mathbf{\Sigma}$', defining formulas begin with a block of existential quantifiers
- For sets in classes denoted with '$\mathbf{\Pi}$', defining formulas begin with a block of universal quantifiers
- The subscript in '$\Box^1_n$' (where $\Box \in \{\mathbf{\Sigma}, \mathbf{\Pi}\}$) denotes n alternations of quantifiers

When we do not allow real parameters, we generate the corresponding familiar *lightface* hierachies of *effective descriptive set theory*:

- the *Arithmetical hierarchy*, which is analogous to the Borel hierarchy, and
- The *Analytical hierarchy*, which is analogous to the Projective hierarchy

In the effective case, by analogy, we take ω for $\mathbb{R}$, recursive functions for continuous functions, hyperarithmetical sets for Borel sets, and analytical sets for projective sets.
In either case, the heart of these hierarchies is that the objects in new classes are defined in terms of those objects defined in earlier classes.

Note a disanalogy, however, between our characterization of the Borel and Projective hierarchies: the former applies to all classes indexed by a countable ordinal whereas the latter applies only to those classes indexed by finite ordinals.

Recall from the introduction to this seminar that, for X a transitive set, a subset $Y \subseteq X$ is "*definable* in $(X, \in)$ with parameters" when, for some first-order formula ϕ and parameters $y_1, ..., y_n \in X$, we have $Y = \{y \in X | (X, \in) \models \phi(y, y_1, ..., y_n)\}$. The set of all such Y we denote here by $Def(X)$.

$L(\mathbb{R})$

Recall that, for some set X, $L_0(X) = TC(X)$ where $TC(X)$ is the smallest transitive Y s.t. $X \in Y$. Note $\omega \subseteq V_\omega$ but $\omega \notin V_\omega$—taking $\mathcal{P}(V_\omega)$ gives us the set of all functions on $\omega \times \omega$ and so we begin with $V_{\omega+1}$. The hierarchy $L(\mathbb{R})$ is accordingly generated as follows:

- $L_0(\mathbb{R}) = V_{\omega+1}$
- $L_{\alpha+1}(\mathbb{R}) = Def(L_\alpha(\mathbb{R}))$
- For limit ordinals λ, $L_\lambda(\mathbb{R}) = \bigcup_{\alpha<\lambda} L_\alpha(\mathbb{R})$

$L(\mathbb{R})$ denotes, for $\alpha \in On$, $\bigcup_\alpha L_\alpha(\mathbb{R})$. The Projective sets are exactly those in $\mathcal{P}(\omega^\omega) \cap L_1(\mathbb{R})$—insofar as we'd like to remain agnostic about the truth of AC in $L(\mathbb{R})$, we take the intersection to avoid assuming that there is a (projectively definable) well-ordering of $\mathbb{R}$ in $L(\mathbb{R})$.

A quick preview of of the relevance of $L(\mathbb{R})$:

Assuming $ZFC+$"there exists a proper class of Woodin cardinals," it follows that the theory of $L(\mathbb{R})$ (and so that of the projective sets) is *invariant under set forcing*. We will discuss this strong form of absoluteness next week.

Lebesgue Measurability

We define Lebesgue measurability together with a measure function μ that maps subsets of ω^ω to $[0,1]$:

- For $s \in \omega^{<\omega}$ a finite sequence with $|s| = n$, $\mu([s]) := \prod_{i=0}^{n} \frac{1}{2^{s(i)+1}}$
- The standard measure-theoretic construction extends μ to all Borel sets via induction on negation and countable union
- For $B \in \mathcal{B}$, B is *Lebesgue null* if $\mu(B) = 0$
  - For arbitrary $X \subseteq \omega^\omega$, X is Lebesgue null if $X \subseteq B$ for some $B \in \mathcal{B}$ such that $\mu(B) = 0$—then we define $\mu(X) := 0$
- For $X \subseteq \omega^\omega$, X is *Lebesgue measurable* if there exists a $B \in \mathcal{B}$ s.t. $X \backslash B \cup B \backslash X$ (i.e., their symmetric difference) is Lebesgue null—then we define $\mu(X) := \mu(B)$

Hence, a set X has this property when it differs from a Borel set by a Lebesgue null set.

For $X \subseteq \omega^\omega$,

- X is *meagre* just in case X is the countable union of nowhere dense sets
- X has the *property of Baire* just in case there exists an open set O s.t. $X \backslash O \cup O \backslash X$ is meagre

Hence, a set X has this property when it differs from an open set by a meagre set—a type of set which has a measure-theoretic analogue in the form of null sets. Both are, in their respective contexts, "negligible." For instance, $\mathbb{Q}$ is a meagre subset of $\mathbb{R}$. Last, recall that the Baire Category Theorem ensures us that every non-empty open subset of ω^ω is non-meagre.

For $X \subseteq \omega^\omega$,

- X is *perfect* just in case: (i) $X \neq \emptyset$, (ii) X is closed, and (iii) X contains no isolated points (i.e., there is no $x \in X$ s.t. $\{x\}$ is open in X)
  - Every perfect set X in a Polish space contains a homeomorphic copy of Cantor space 2^ω and therefore has the cardinality of the Cantor set 2^ω
- X has the *perfect set property* just in case either: (i) X is countable or (ii) there is a $Y \subseteq X$ s.t. Y is perfect

The perfect set property is of interest insofar as the sets of reals with this property also satisfy *CH* (as can be seen from the second clause).

These three properties are called the *regularity* properties of sets and are diagnostic of the "well-behavedness" of subsets of ω^ω studied in descriptive set theory.

Qualitatively speaking, we can view such sets as arrayed along a spectrum of "well-behavedness"—at one extreme, the (relatively) simple sets defined by application of countable union and complementation from open sets at the base of the Borel hierarchy. And, at the other extreme, the (relatively) complex sets defined by arbitrary first-order formulas from $V_{\omega+1}$ at the base of $L(\mathbb{R})$.

The increasing (naive, qualitative) complexity of sets along this spectrum derives from the two clauses invoked in the definitions of the hierarchies involved:

- The complexity of the subsets of ω^ω used as the basis of the inductive definitions of each hierarchy. I.e.:
  - For Borel, we use the open sets
  - For Projective, we use the analytic sets, which are (relatively) more complex than the open sets
  - For $L(\mathbb{R})$, we use $V_{\omega+1}$, which is (relatively) more complex than the analytic sets

and...

- The complexity of the permissible operations used in the "step" of the inductive definitions of each hierarchy. I.e.:
  - For Borel, we apply a simple generalization of a boolean operation (i.e., countable union and complementation) to previously-defined sets
  - For Projective, we additionally apply what is essentially an existential quantification (i.e., projection)—an operation more complex than boolean—to previously-defined sets
  - For $L(\mathbb{R})$, we additionally apply operations defined from formulas of arbitrary first-order complexity—operations more complex than projection—to previously-defined sets

Some philosophical questions then emerge to motivate metamathematical investigation of the material presented so far:

- Is there an obvious distribution of the regularity properties across this spectrum? If so, what is the distribution?
- Is there relevance of this distribution? If so, what is the relevance?
- Does this distribution seem to be strongly correlated with the logical strength of the assumptions required to prove those properties hold? If there seems to be a strong correlation, why does it exist?
- What sorts of properties of (relatively simple) sets in ω^ω should mathematicians expect to hold of (relatively complex) sets in ω^ω? Why should we expect them to hold in this way?

Here is a timeline of the discovery of regularity properties of definable sets of reals. Assume *ZFC*.

- 1883 (Cantor and Bendixson): $\mathbf{\Pi}^0_1$ perfect set property
- 1903 (Young): $\mathbf{\Pi}^0_2$ perfect set property
- 1916 (Aleksandrov): $\mathbf{\Delta}^1_1$ perfect set property
- 1917 (Luzin and Suslin): $\mathbf{\Delta}^1_1$ Lebesgue measurability, property of Baire
- 1917 (Luzin and Suslin): $\mathbf{\Sigma}^1_1$ perfect set property, Lebesgue measurability, property of Baire
- 1917 (Luzin and Suslin): $\mathbf{\Pi}^1_1$ Lebesgue measurability, property of Baire

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- 1938 (Gödel): Suppose $V = L$, then there are $\mathbf{\Pi}^1_1$ without perfect set property
- 1938 (Gödel): Suppose $V = L$, then there are $\mathbf{\Sigma}^1_2$ non-Lebesgue measurable, without property of Baire
- 1965 (Solovay): Suppose there is a measurable cardinal, then every $\mathbf{\Sigma}^1_2$ perfect set property, Lebesgue measurability, property of Baire

Moschovakis writes of these last results that, "the logicians entered the picture in their usual style, as spoilers." In particular, the last two results demonstrate that the question of the whether the regularity properties hold of projective sets at $\mathbf{\Sigma}^1_2$ (and beyond) is independent of ZFC.

So, at the $\mathbf{\Sigma}^1_2$ level and beyond, the question of whether the rest of the projective sets and the sets in $L(\mathbb{R})$ have the regularity properties is completely underdetermined by ZFC. We can now refine some of our philosophical questions:

- Are there mathematically well-justified principles independent of ZFC that can decide whether $\mathbf{\Sigma}^1_2$ sets have the regularity properties?
- How about the sets in $\mathcal{P}(\omega^\omega) \cap L_1(\mathbb{R})$?
- And even $L(\mathbb{R})$?
- If there are such principles, how much logical strength do they add to ZFC? What kind of logical strength do they add?
- Do we have reason to accept them? If so, are they axioms?

Starting in the 1950s and 1960s, descriptive set theorists began to investigate the consequences of asserting that there exist winning strategies for infinite games of perfect information. These games are "played" over (representations of) of reals in the underlying space. It was discovered that the existence of winning strategies for these games, or *determinacy*, implies that their payoff sets have all of the regularity properties.

The *Axiom of Determinacy*, AD, asserts that every set is determined. Because AD is itself inconsistent with AC, instead restrictions of AD that are consistent with AC are typically employed—namely, restrictions to the definable sets in the Borel hierarchy, the Projective hierarchy, and those in $L(\mathbb{R})$.

Moreover, many have attempted to address the metamathematical barriers that stymied the classical theory through an approach that integrates *principles of definable determinacy* and *large cardinal hypotheses*.

Next week:

- Begin with infinite games of perfect information, strategies for these games, and definable determinacy.
- Then discuss metamathematical issues pertinent to $\mathbf{\Delta}^1_1$ Determinacy, Projective Determinacy (PD), and $AD^{L(\mathbb{R})}$.
- Last, examine the case supporting Steel and Martin's 1985 proof of PD from $ZFC+$"there exist infinitely-many Woodin cardinals."

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