Generalized Models: Opportunism, or Courage?

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Henkin's models for higher-order logic widely used technique, but their status a matter of dispute. Ongoing work with Hajnal Andréka, Nick Bezhanishvili & Ístvan Németi on the scope and justification of the method. We look at general models in terms of 'the intended models', calibrating proof strength, algebraic representation, lowering complexity of core logics, and absoluteness. Our aim is general perspectives on design of generalized models, and connections between these. We state new results in the latter vein, while exploring the scope of generalized models on a new benchmark: fixed-point logics of computation.

1 General models for second-order logic

Second-order logic: syntax of first-order logic, plus second-order quantifiers $\exists X\varphi$, $\forall X\varphi$. *Standard models* $\boldsymbol{M} = (D, I)$: domain of objects D, interpretation I for predicate constants.[1] *Assignments s* send individual variables to objects, predicate variables to real predicates.

$\boldsymbol{M}, s \models \exists X\varphi$ iff *there is some predicate P of suitable arity with* $\boldsymbol{M}, s[X:=P] \models \varphi$

Now we take control over the predicates, instead of relying on set theory. *General model* $(\boldsymbol{M}, \boldsymbol{V})$: $\boldsymbol{M}$ a standard model, $\boldsymbol{V}$ a non-empty family of predicates on the domain D of $\boldsymbol{M}$.

$\boldsymbol{M}, s \models \exists X\varphi$ iff *there is some predicate P of suitable arity* ***in V*** *with* $\boldsymbol{M}, s[X:=P] \models \varphi$

Theorem Second-order logic is recursively axiomatizable over general models.

This can be shown by adapting a Henkin-style completeness proof for first-order logic.

Alternative: *translation into two-sorted first-order logic*, with sorts \`objects' and \`predicates' (unary, for convenience). The domains are connected by a predicate $E\,xp$ saying that object x belongs to predicate p. All clauses are homomorphic, and for atoms we set:

Xy goes to $E\,yX$

In addition, we make predicates behave like set-theoretic predicates in general models:

Extensionality (EXT) $\forall pq\,(p=q \leftrightarrow \forall x(E\,xp \leftrightarrow E\,xq))$

Fact For all second-order formulas φ, φ is valid on all general models iff $\tau(\varphi)$ follows from *EXT* on all models for the two-sorted first-order language.

Corollary The validities of second-order logic over general models are axiomatizable.

Modulations: * drop extensionality (have real intensional predicates), * add stronger existence principles on predicates. E.g., reflecting existential instantiation $\varphi(\psi) \to \exists X\,\varphi(X)$:

Comprehension $\exists X\,\forall y\,(Xy \leftrightarrow \varphi)$ for arbitrary second-order formulas φ

Many other constraints: e.g., the 'separation axiom' $\forall xy\,(x=y \leftrightarrow \forall p(E\,xp \leftrightarrow E\,yp))$.

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[1] The very term \`standard model' prejudges the issue of whether *other* models might also be natural.

The full complexity of second-order logic resides in its universal fragment:

Fact A second-order formula is valid on standard models iff its first-order translation $\tau(\varphi)$ follows from EXT plus the second-order formula $\forall X\ \exists p\ \forall y(Xy \leftrightarrow E\,yp)$.

Many extensions of this line of thinking: type theories, etc.

Hard-work results: decidability key *fragments* of second-order logic (Rabin's Theorem).

2 First round: objections and rebuttals

Criticisms * Ad-hoc, *intended semantics* is standard models. * Circularity: impredicativity of comprehension. * Cheapness: the completeness proof yields nothing new.
Responses * Philosophy: predicates or propositions are primitive, beyond set-theoretic extensions, * Mathematics: set-theory is not the norm, many primitives on a par (geometry), category theory: control morphisms, * Logic: standard models use a set-theoretic magic wand, avoid the real work.

3 Logical perspectives on controlling predicates

Proof Theory Parsimony in definable predicates (constructivism, reverse mathematics). Hilbert-Bernays Completeness Theorem: First-order logic complete for models with Δ^0_2–predicates on natural numbers. Recursive will not do (Vaught): must stay 'in between'. Many other results in the same vein (FO-definable predicates not sufficient, need 'junk'.)
New structure Controlling predicates reveals important new structure. Objects and types as duals in Chu Spaces with Chu morphisms. Dependence games in IF logic (existential second-order logic of Skolem functions). Standard models: truth = existence of winning strategy for Verifier, non-axiomatizable logic. Henkin models: determinacy lost, probabilistic equilibria remain. Richer mathematics than original IF logic: weakness is wealth.

4 General models and algebraic representation

* Algebraic representation theorems produce general models. Jónsson-Tarski Theorem: modal BAOs representable as subset algebras with interior operation on frames *(W, R)*. Happens all the time. Recent example: Holliday on Humberstone's possibilities models.

* Virtue: the right approach to obtaining a full category-theoretic duality.

* Issue: JT representation produces special 'descriptive frames' (separating, compact).

* Aside: JT also uses all ultrafilters as points, one more undesirable set-theoretic feature?

* Issue: so, why do we like modal completeness theorems with respect to frame classes? (Self-criticism: maybe because modal correspondence theory is biased toward frames?)

Conclusion In algebraic perspective, general models are the natural mathematical objects.

5 General models, lowering complexity, core logics

* General models lower complexity of second-order logic to first-order (RE, undecidable).

Theme: what is the core calculus of a logic? No need to stop at undecidable.

* Hunt for 'standard' set-theoretic features that can be modulated. Tarski semantics: Models $\boldsymbol{M} = (D, I)$ with objects D plus interpretation map I assigning predicates over D to predicate symbols in the language. The crucial truth condition for the existential quantifier:

$$\boldsymbol{M}, s \models \exists x\varphi \quad \text{iff} \quad \textit{there is some } d \textit{ in } D \textit{ with } \boldsymbol{M}, s[x:=d] \models \varphi$$

Has some set-theoretic 'wrappings' plus a fateful hidden object: the full function space D^{VAR}.

* The only thing needed for Tarski's famous recursion is abstract modal universe of states:

$$\boldsymbol{M}, s \models \exists x\varphi \quad \text{iff} \quad \textit{there is some } s' \textit{ with } sR_x s' \textit{ such that } \boldsymbol{M}, s' \models \varphi$$

Decidable minimal modal logic for the quantifiers covers a lot of first-order reasoning.

* Next stage: *general assignment models* $(\boldsymbol{M}, V)$, with $\boldsymbol{M}$ a standard first- order model (D, I), and V a set of maps from individual variables to objects in D. New truth definition:

$$\boldsymbol{M}, s \models \exists x\varphi \quad \text{iff} \quad \textit{there is some } d \textit{ in } D \textit{ such that } s[x:=d] \in V \textit{ and } \boldsymbol{M}, s[x:=d] \models \varphi.$$

Gaps' in the space of all assignments encode possible *dependencies* between variables.

Theorem First-order logic over general assignment models is decidable.

* *Fine-structure of predicate logic*. Further axioms encode existence assumptions.

Fact A modal frame satisfies the axiom $\exists x \forall y\, Rxy \rightarrow \forall y\, \exists x\, Rxy$ iff its transition relations R_x, R_y satisfy for all states s, t, u: *if* $s R_x t$ *and* $s R_y u$, *then there is a* v *with* $t R_y v$ *and* $u R_x v$.

Fact Adding confluence to general assignment logic makes the logic undecidable.

* *Richer languages.* Generalized models suggest richer logical languages. *Polyadic quantifiers* $\exists\boldsymbol{x}\varphi$ where $\boldsymbol{x}$ is a finite tuple of variables in terms of available `simultaneous re-assignment'.

Theorem FOL with polyadic quantifiers is decidable over general assignment models.

Other examples: extension modality between assignment models. Standard $\exists x$ is $\Diamond\exists x$.

$$\boldsymbol{M}, \boldsymbol{V}, s \models \Diamond\varphi \quad \text{iff} \quad \textit{there is some } \boldsymbol{V'} \supseteq \boldsymbol{V} \textit{ with } M, V', s \models \varphi$$

Is this logic RE? Generalized models may change the whole design of languages and logics.

* *Guarded Fragment*: only quantifiers $\exists\boldsymbol{y}\, (G(\boldsymbol{x}, \boldsymbol{y}) \wedge \varphi(\boldsymbol{x}, \boldsymbol{y}))$, where $\boldsymbol{x}, \boldsymbol{y}$ are tuples of variables, G is an atomic predicate, and $\boldsymbol{x}, \boldsymbol{y}$ are the only free variables occurring in $\varphi(\boldsymbol{x}, \boldsymbol{y})$:

Theorem FOL on general assignment models translates into GF over standard models.

Full language over generalized models reduces to syntactic fragment over standard models.

Conclusion Generalized models can control complexity, and lead to new languages.

6 Generalized models for fixed-point logics

* Fixed-point logics count along numbers (transitive closure) or ordinals (LFP(FO)). Method in literature: drop 'standard counting', and count along non-standard models of arithmetic, added as part of the models. (E.g., transitive closure can now have infinite sequences.)

Theorem (Budapest) Predicate dynamic logic with non-standard natural numbers is RE.

Moreover, the logic contains all the usual principles actually used in program semantics.

- But also Henkin model approaches to modal fixed-point logics.

Theorem (London) LFP(FO) over generalized models with comprehension for fixed-point formulas is equivalent to the modal μ–calculus.

How are the two methods related? Conjecture: first-approach special case of the second.

Third approach: there are also general-assignment versions of LFP(FO) semantics.

7 General models and absoluteness

To be added (work by Barwise, Manders, Feferman, Bonnay, Hamkin, Andréka, Németi). Viewing Henkin models as the unique set-theoretically absolute version of standard models.

8 Relating different perspectives on general models

* *How are our general perspectives related?* Do approaches reduce, or at least, combine?

Theorem The guarded fragment of second-order logic is decidable.

* *Richer languages for generalized models?* E.g., do Henkin's models suggest extensions of the second-order language for predication? (Example: inclusion modality once more.)
* *Trade-offs generalized semantics and language fragments.* In a sense the two-sorted view, is such a fragment reduction. But also: explain low complexity of well-known SOL fragments as form of generalized semantics? (Simple case: what of propositional PDL?)
* Are there yet other fruitful general methods for generalizing existing semantics? One line: *nonstandard truth conditions*. Example 1: lower complexity of propositional logic to Ptime: new models, or: change semantics of disjunction? Example 2: `bisimulation quantifiers' $<p>\varphi$ saying in models ***(M**, s)* that there exists a bisimulation, for the whole language except for the proposition letter *p*, between ***(M**, s)* and some model ***(N**, t)* such that $\varphi$ is true in ***(N**, t).* One can think of this as a `tamed form' of second-order quantification over the property *p*, which leaves modal logic decidable and even yields striking model-theoretic properties such as uniform interpolation. A similar move would be possible for second-order logic.

9 Conclusion

Many forms of generalizing semantics (some of course ad-hoc, but many with a systematic thrust). Interesting inventory, compare, combine. General issue: what is the real idea behind a logical system, and what are its accidental 'wrappings' when it was first proposed?

Coda: what about the Henkin completeness proof itself? How to remove the standardness?