Fixing Frege's Set Theory

Flash Sheridan

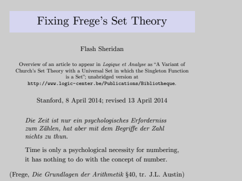
Overview of an article to appear in *Logique et Analyse* as "A Variant of Church's Set Theory with a Universal Set in which the Singleton Function is a Set"; unabridged version at
`http://www.logic-center.be/Publications/Bibliotheque`.

Stanford, 8 April 2014; revised 13 April 2014

Die Zeit ist nur ein psychologisches Erforderniss zum Zählen, hat aber mit dem Begriffe der Zahl nichts zu thun.

Time is only a psychological necessity for numbering, it has nothing to do with the concept of number.

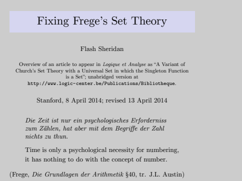
(Frege, *Die Grundlagen der Arithmetik* §40, tr. J.L. Austin)



Speaker Notes

My thanks to Daniel Isaacson, Robin Knight, and Rolf Suabedissen of the University of Oxford, Michael Beeson of San Jose State University, and Solomon Feferman and Ed Zalta of Stanford University for feedback on earlier versions of this talk. Written in partial fulfilment of the requirements for the degree of Doctor of Philosophy at the University of Oxford.

1. Today I'm going to talk about the rewritten version of my doctoral thesis on Church's Set Theory with a Universal Set and my variant of it; mainly the philosophical motivation for, and problems with, both. The actual proofs tend to get quite tedious quite quickly; the details are at `http://www.logic-center.be/Publications/Bibliotheque`.



2. The trajectory is from the philosophical motivation, to the foundations of arithmetic via definition by abstraction, to set theoretic axioms, to relative consistency proofs. Perhaps details of my sequence of restricted equivalence relations generalizing equinumerosity, if there's time and interest.

3. So I'll start with a quotation from Gottlob Frege on mathematical Platonism, or at least the independence of mathematical objects from time, which is all the Platonism we'll need today.

Abstract 1

Fixing Frege's Set Theory: Mathematical Platonism and Church's Set Theory with a Universal Set

I discuss Alonzo Church's Set Theory with a Universal Set, together with my variant of it, and what I believe to be the philosophical motivation behind—and problems with—the theory. It allows the definition of *some* cardinal numbers by abstraction from the concept of equinumerosity, since it proves the existence of the Frege-Russell cardinal for every well-founded set.

This is still, I maintain, the most satisfying answer to the question, "What is a number?", and is still the best available (albeit partial) explanation of the unreasonable effectiveness of mathematics. It brings the Neo-Fregean program somewhat closer to Frege's original intent, by avoiding the postulation of Hume's Principal.

Abstract 2

Church's concluding remark, and his later unpublished and abandoned work, attempted to unify his approach with Quine's *New Foundations*. I maintain that Quine's comprehension schema is Platonistically unacceptable, and that the theory of cardinalities in *New Foundations* is mathematically regrettable, due in part to the apparently arbitrary denial of the existence of the singleton function.

My variant of Church's theory proves the existence of the singleton function, via a modification of Church's generalization of equinumerosity to a sequence of equivalence relations. A natural extension of my theory, with an eye to Feferman's desiderata for the foundations of category theory, would lead to a variant of the Russell paradox.

Church sketched a construction to prove the equiconsistency of his theory with ZFC, though he apparently abandoned the actual proof. I discuss my simplification of his technique for the consistency proof of my own system, and present a picture of my variant of his sequence of equivalence relations.

Platonism versus Time

ταῦτα δὲ πάντα μέρη χρόνου, καὶ τό τ᾽ ἦν τό τ᾽ ἔσται χρόνου γεγονότα εἴδη, ἃ δὴ φέροντες λανθάνομεν ἐπὶ τὴν ἀίδιον οὐσίαν οὐκ ὀρθῶς. λέγομεν γὰρ δὴ ὡς ἦν ἔστιν τε καὶ ἔσται, τῇ δὲ τὸ ἔστιν μόνον κατὰ τὸν ἀληθῆ λόγον προσήκει, τὸ δὲ ἦν τό τ᾽ ἔσται περὶ τὴν ἐν χρόνῳ γένεσιν ἰοῦσαν πρέπει λέγεσθαι—κινήσεις γάρ ἐστον

And these are all portions of Time; even as "Was" and "Shall be" are generated forms of Time, although we apply them wrongly, without noticing, to Eternal Being. For we say that it "is" or "was" or "will be," whereas, in truth of speech, "is" alone is appropriate to It; "was" and "will be," on the other hand, are properly said of the Becoming which proceeds in Time, since [both of] these are motions...

(Plato, *Timæus* 37E, tr. R.G. Bury (Loeb), corrected slightly)

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1. `http://www.perseus.tufts.edu/hopper/text?doc=Perseus%3Atext%3A1999.01.0180%3Atext%3DTim.%3Asection%3D37e`
2. This was the best quotation from Plato I was able to find on this part of what we now call mathematical Platonism. (There's a more relevant quotation in the *Republic*, but it's less clear, [as is common with Plato], since on the surface it's about planets rather than abstract objects.)
3. But I think it shows that Plato, too, disapproved of applying time to abstract objects. So to that extent, I think it's fair for me to use the term "mathematical Platonism" at least in this context. Adherence to Plato's ideas about anything else is not required; the term "Pythagoreanism" might do as well.

What is a Number?

Frege's *Grundgesetze der Arithmetik* [1893 & 1903] is still the most satisfying answer to the question, "What is a cardinal number?"

Hume's Principle

The number belonging to the concept (i.e., predicate) F is identical with the number belonging to the concept G if the concept F is equinumerous with the concept G (i.e., there is a 1-1 onto mapping).

Definition by Abstraction: Frege-Russell Cardinal

The *number of a concept* Φ is the extension of the concept "equinumerous to Φ."

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[Following Heck [2013], for informal exposition I'm eliding the difference between the formulations in the *Grudgesetze* and *Grudlagen*, and between concepts and their extensions.]

Virtues of the *Grudgesetze*

- Natural definition
- Rigorous development: more so than almost any current mathematics outside of automated deduction. (Full proofs, *not* sketches.)
- Explains the unreasonable effectiveness of mathematics: application is part of the foundation.
  - Likewise for real numbers, which I won't discuss here.

The Problem: The Russell Paradox

Naive comprehension is inconsistent

- $\{x|x\notin x\} \in \{x|x\notin x\} \iff \{x|x\notin x\} \notin \{x|x\notin x\}$
- The Zermelo-Frænkel (and arguably Cantorian) solution is to ignore the universe, and any other sets that are too large, and use only iteratively-constructed sets.
- The Neo-Fregeans give up on Frege's set theory, and look at his (largely separable) development of arithmetic from Hume's Principle. (Burgess, *Fixing Frege*)

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Ignoring the universe is not the mandatory solution to the Russell Paradox, though a lot of people get this wrong, e.g., Jech's *Set Theory* on page 4. When a conjunction is disproved, you can't just pick which conjunct to deny. (Quine has more to say about that...)

Restricted Comprehension Schemes

- Quine's *New Foundations:* arbitrarily restricting naive comprehension. The arbitrariness is clearest in Hailperin's finite axiomatization—the singleton function is mysteriously excluded.
- Holmes may recently have proven NF's consistency, but I am not satisfied with his quasi-nominalist philosophical justification.
- Due to the lack of the singleton function, its cardinal numbers have *ugly* properties. E.g., the universal set is strictly larger than the set of all singletons (Frege's 1), and so on, all the way down.

Set Theories with a Universal Set and Frege Cardinals

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See [Holmes 1998], chapter 8 for the quasi-nominalism, and `http://cs.nyu.edu/pipermail/fom/2012-November/016758.html` & `http://www.cs.nyu.edu/pipermail/fom/2013-October/017663.html` for progress on the consistency proof.

Restricted Comprehension Schemes

- Oberschelp's theory *does* have the singleton function as a set; the restriction to his comprehension schema seems to be what will work for his model (and for interpretations like Church's and mine.)
- Not clear that his proof is actually complete; I feel that Oberschelp's work deserves far more attention than it has received, but the presentation is difficult, and part of the consistency proof is a reference to another paper with a different formalism.

Platonist Objection to Comprehension Schemes

- A whole-hearted Platonist should object to *any* first-order comprehension schema: it uses a temporal, iterative, metaphor, constructing a new set given a property. Mathematical objects are not constructed; they exist (or do not) outside of time.
- The particular example in the Russell Paradox amounts to a claim that two simple set-theoretic operations (adjoining and subtracting a single object) have implausible fixed points, which also happen to be the same.
- This claim was the point to my undergraduate thesis [1982]; see also Dummett [1991] and Forster & Libert [2011].
  - See Skala [1974] for a set theory which accepts the *reductio ad absurdam* and fails to have even some crucial singletons.

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I am grateful to Edward Zalta for pointing out the need to restrict the objection to first-order comprehension. Cp. a remark by Church in a paper on which he was working at around the same time: "To avoid impredicativity the essential restriction is that quantification over any domain (type) must not be allowed to add new members to the domain, as it is held that adding new members changes the meaning of quantification over the domain in such a way that a vicious circle results." "Comparison of Russell's Resolution of the Semantical Antinomies with that of Tarski," *The Journal of Symbolic Logic* volume 41, Number 4, Dec. 1976.

Church's 1971 Set Theory: Motivation or Lack Thereof

- Church never revealed his philosophical motivation.
- New axioms fit his apparent Platonism:
  - K: Complements.
  - L_j: $j = 1$ gives Frege cardinals, at least for well-founded sets.
    - And generalizations thereof, j-cardinals for finite j.

Church's Basic Axioms

3. Basic axioms. As *basic axioms of set theory* we take the following axioms A–J. (The number of axioms is in fact infinite, since A, H, and I are axiom schemata rather than single axioms.)

A. *Substitutivity*. $y = z \supset . \mathbf{A}_y \supset \mathbf{A}_z$.

B. *Extensionality*. $x \in y \equiv_x x \in z \supset y = z$.

C. *Pair set*. $(\exists u) . x \in u \equiv_x . x = y \vee x = z$.

D. *Sum set*. $(\exists u) . x \in u \equiv_x (\exists y) . y \in z . x \in y$.

E. *Product set*. $y \in z \supset (\exists u) . x \in u \equiv_x . y \in z \supset_y x \in y$.

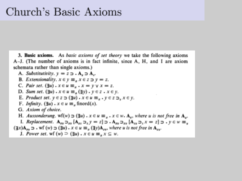
F. *Infinity*. $(\exists u) . x \in u \equiv_x \mathrm{finord}(x)$.

G. *Axiom of choice*.

H. *Aussonderung*. $\mathrm{wf}(w) \supset (\exists u) . x \in u \equiv_x . x \in w . \mathbf{A}_x$, *where u is not free in* $\mathbf{A}_x$.

I. *Replacement*. $\mathbf{A}_{xy} \supset_{xy} [\mathbf{A}_{xz} \supset_z y = z] \supset . \mathbf{A}_{xy} \supset_{xy} [\mathbf{A}_{zy} \supset_z x = z] \supset . y \in w \equiv_y (\exists x)\mathbf{A}_{xy} \supset . \mathrm{wf}\,(w) \supset (\exists u) . x \in u \equiv_x (\exists y)\mathbf{A}_{xy}$, *where u is not free in* $\mathbf{A}_{xy}$.

J. *Power set*. $\mathrm{wf}\,(w) \supset (\exists u) . x \in u \equiv_x x \subseteq w$.



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Don't worry about the details of Church's formalism; explanations are on following slides.

The Basic Axioms

- Old, iteratively constructed, sets are the same.
- The Basic Axioms (A–J) are variants of ZFGC (Zermelo-Fraenkel plus Global Choice) axioms, but three of them are restricted to well-founded sets. (Four in my system.)
  - Perhaps hypocritical, but highly convenient, both for consistency proofs and potential use of the theory.
  - Perhaps unwisely, I developed much of the consistency proof in my variant of the Basic Axioms, and (to answer a question from the examiners) in more generality than necessary, and with only limited use of Choice.

Basic Axioms Restrictions

- Basic Axioms H (Separation) and I (Replacement) must be restricted to well-founded sets, by the philosophical argument above, and to avoid the Russell Paradox.
- Because of limitations in the consistency proof technique, Axiom J (Power Set) is also restricted to well-founded sets in Church's and my theories, but not Mitchell's.
- Sum Set (D) is also restricted to well-founded sets in Mitchell's and my systems, but not Church's. Its truth in Church's interpretation seems to depend on a limitation of his equivalence classes.

Limitations

- Church's (and my) consistency proof technique will only provide generalized Frege cardinals for well-founded sets, and those j-equivalent to them.
- Even in improved theories, lack of set functions witnessing equinumerosity might make such cardinals defective. (Cp. perversities caused in *New Foundations* by lack of singleton function.)
- A natural extension of my (not Church's) j-equivalence classes, when *not* restricted to well-founded sets, leads to a variant of the Russell Paradox, Flash's Paradox of the set of all non-self-membered sets equinumerous to the universe.

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1. I was led to the paradox when thinking about the usefulness of set theories with a universal set for category theory. Not that I understand category theory; I was just considering the desiderata in [Feferman 2006].
2. The paradox seems to require both removing the restriction of the equivalence class axioms to well-founded sets, and the details of my equivalence relations, in particular their ability to code the membership relation.

Church's j-Equivalence

- Church's j-equivalence is a complicated definition by recursion; he never seems to have done anything with it.
  - 1-equivalence is ordinary equinumerosity.
  - 3-equivalence seems to be order and group isomorphism, suitably formalized.
- The details of my sequence of restricted equivalence relations, j-isomorphism (inspired by Aczel), are relegated to an appendix. The main goal was for the singleton function to be the union of a small number of 2-isomorphism classes.

Church's Interpretations

- Church refers to his construction for a consistency proof as a model, but it's actually just an interpretation of the axioms in ZFGC.
- The published article describes the interpretation, with j-equivalence classes up to an arbitrary finite m, but omits the proof, calling it "straightforward but (if $m > 0$) laborious."
- In 1985 Church sent me notes taken at a lecture in which he presented the proof for $m = 0$; i.e., with the axiom of complementation, but without j-equivalence classes.

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I.e., purely a syntactic construction, converting any proof of a contradiction in his theory into a proof of a contradiction in the base theory.

Church's Non-Proof

- He never published a proof with equivalence classes, and in 2012 at the Church Archives at Princeton I found what seems to be an abandoned and incomplete draft.
- He later worked on multiple versions of more complicated and less elegant set theories with a universal set, but never published. These seem to have been attempts at what he described in the concluding words of his published article as "a synthesis or partial synthesis of ZF and Quine set theory."

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I found the draft shortly before the archives closed, and I was later denied permission to obtain a copy.

Differences from Church's Construction

- My construction is considerably simpler than Church's, at the cost of using urelements in the style of Oberschelp (and more recently William Brian), and using non-primitive notation in the axioms.
  - Church uses a deliberately ugly form of m-tuple to represent the new sets, to avoid collisions.
- Considerable simplification is obtained by generalizing the equivalence relations to j=0 (the universal relation) and j=ω (equality), which allows for a uniform treatment of complementation and an additional component needed in Church's construction.

Representatives

- The interpretation relies heavily on the notion of a unique representative, j-rep(x), in the base theory for each j-equivalence class i.e., for any x, a unique j-rep(x) is selected which is j-isomorphic to it.
- This could be chosen either as the least member, using Global Choice, or worked around with Scott's Trick; I use the former.

My Representation of the New Sets

- The new membership relation extends the old one. The new sets are represented by urelements, tagged with a sequence of length $\omega+1$, conventionally represented ($L^0 \ldots L^j \ldots L^\omega$), L for short.
- x is a member of a new set (old urelement, tagged by L) if there are an *odd* number of j's such that j-rep(x) is in L^j.
- The idea is that we use urelements to provide only the news sets required by the axioms, plus certain combinations of them, under union and symmetric difference.
  - *Potemkin Village Criticism:* We only provide sets where someone might look for them. Thus nothing like general closure under Cartesian product, nor general existence of mappings.

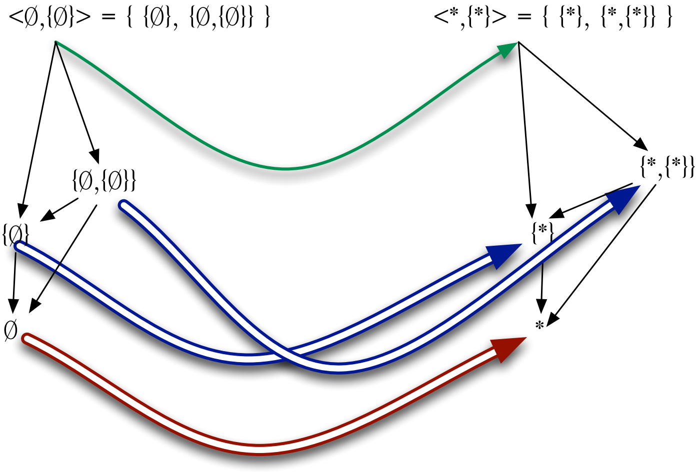
More New Sets

- Thus the universal set is the urelement with tag ({0-rep(Ø)} Ø ... Ø), since everything has the same 0-rep, and 1 is odd.
- The set of all pairs (at least for a normal base model) is tagged by (Ø {1-rep(2)} Ø ... Ø).
- The singleton function will be tagged by (Ø Ø {2-rep(<Ø,{Ø}>)} Ø ... Ø).
  - Details later.
- The complement of ω is tagged by ({0-rep(Ø))} Ø ... ω-rep“ω).

Appendix: My Sequence of Restricted Equivalence Relations

2-Isomorphism Diagram

$$F: \langle \emptyset, \{\emptyset\} \rangle \Bumpeq^2 \langle *, \{*\} \rangle$$



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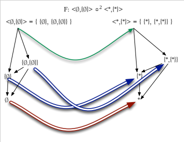
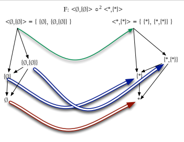


1. Note that this slide may not be comprehensible until you have understood the textual definition two slides hence, and, of course, *vice versa.*
2. Here's a picture, in the style of Aczel, of what a 2-isomorphism (F) looks like, in this case taking (from the bottom up) the empty set to an unspecified object "Star", and (working upwards) consequently maps the singleton of the empty set to the singleton of Star, and so on, up to taking the ordered pair <empty set, singleton of the empty set> to <Star, singleton of Star>. This, of course, is one member of the Singleton Function.

2-Isomorphism Diagram



3. The lowest mapping arrow is in red, indicating that it may lack a property required of the arrows higher up. (Clause 4, below.)

Preliminary Definitions

- $y \in^{\mathbf{j}} a \equiv_{df} \exists f.\ \exists c.\ \text{maps}(f, j+1, c)\ \ \&\ \ f`0 = a\ \ \&\ \ f`j = y\ \ \&\ \ \forall k{\in}j.\ f`k{+}1 \in f`k.$
  Read "y is a member at **level j** of a."
  - Define $y \in^{\mathbf{0}} a \equiv_{df} y = a$; this is odd but convenient for the j^{th} cumulative union, defined below.
- Define $y \in^{<\mathbf{j}} a \equiv_{df} \exists k.\ 0 \leq k < j.\ \ y \in^{k} a.$
- Define $y \in^{\leq \mathbf{j}} a \equiv_{df} \exists k.\ 0 \leq k \leq j.\ \ y \in^{k} a.$
- Define $\Xi^{\mathbf{j}} a =_{df} \{y \mid y \in^{\leq \mathbf{j}} a\}$, for $0 \leq j < \omega$.
  Read "the j^{th} *cumulative union* of a." This is a class abstract; it will have to be proved to be a set before making use of it.

Definition of j-Isomorphism

Define, for $j \leq 1 < \omega$, $a \eqcirc^{j} b \equiv_{df} \exists F$:

- (1) $SET(\Xi^{j}a)$ & $SET(\Xi^{j}b)$ &
- (2) $maps_{1\text{-}1}(F, \Xi^{j}a, \Xi^{j}b)$ &
- (3) $F`a = b$ &
- (4) $\forall y \in^{<j} a.\ F`y = F``y$

Read "a is **j-isomorphic** to b." When F is known, I will also write "F: $a \eqcirc^{j} b$."
For convenience, define $\eqcirc^{0}$ as the universal relation which holds between any two objects, and $\eqcirc^{\omega}$ as equality.
Note that proving this an equivalence relation is non-trivial, and not necessarily true for large ill-founded sets in this sort of interpretation.

j-Isomorphism Examples

- Two sets are 1-isomorphic iff they're equinumerous and both (or neither) are self-membered:
  - $\forall a,b.\ \text{non-empty}(a) \Rightarrow a \Bumpeq^1 b \equiv .\ a \approx b \ \ \& \ \ (a \in a \equiv b \in b)$
- Something is in the singleton function iff it's 2-isomorphic to $<\emptyset, \{\emptyset\}>$:
  - $\forall b.\ <\emptyset, \{\emptyset\}> \Bumpeq^2 b \equiv \exists d.\ b = <d, \{d\}>$

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