Guarded Negation

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Based on joint work with Luc Segoufin, Vince Barany, and Martin Otto
(STACS 11, ICALP 11, VLDB 12)

Decidable Fragments

- Decidable fragments of first-order logic (FO):
  - Restricting quantifier alternation (e.g., Bernays, Schoenfinkel, 1928)
  - Restricting the number of variables (e.g., Scott 1962, Mortimer 1975)
  - Restricted quantification patterns (Andreka, vBenthem, Nemeti 1998)
  - Restricting the use of negation

Explaining the Good Behavior of Modal Logic

- “Why is modal logic so robustly decidable?” (Vardi 1996)
- “What makes modal logic tick?” (Andreka, van Benthem, Nemeti 1998)
- Good properties of modal logic:
  - Finite model property
  - Tree model property
  - Decidability
  - Craig interpolation
  - ...

- The modal fragment:
  - $\phi(x) := P_i(x) \mid \phi(x) \wedge \psi(x) \mid \neg\phi(x) \mid \exists y(Rxy \wedge \phi(y))$

- **Guarded Fragment (GF)**: restricted patterns of quantification. [Andreka, Van Benthem, Nemeti 1998]
  - A very successful idea, that gave rise to a beautiful and rich body work.

- **Here** another avenue: restricting the use of negation.
  - Unary negation (UNFO): allow only $\neg\phi(x)$ [tC & Segoufin 11]
  - Guarded negation (GNFO): allow also $G(\mathbf{x}) \wedge \neg\phi(\mathbf{x})$ [Barany, tC & Segoufin 11]

Why Restricted Negation?

- A new way of looking at modal logic
- Gives rise to interesting new decidable fragments
- The idea that “negation is dangerous” is prominent in DB theory, and most database queries in practice use a limited form of negation.

Unary Negation

- Unary Negation FO (UNFO):
  - $\phi ::= R(\mathbf{x}) \mid x = y \mid \phi \land \phi \mid \phi \lor \phi \mid \exists x \phi \mid \neg\phi(x)$
  - Only allow negation of formulas in one free variable.
  - NB. The universal quantifier is treated as a defined connective.
- Fixed-Point Extension (UNFP):
  - $\phi ::= \ldots \mid [\mathrm{LFP}_{Z,z}\, \phi(Z,\mathbf{Y},z)](x)$ (where Z occurs only positively in ϕ)
  - NB. No first-order parameters (i.e., only one free first-order variable).

Examples

- Every node lies on a 3-cycle (in <u>UNFO</u> but not in GF or FO^2)

  $\forall x \exists yz.(Rxy \wedge Ryz \wedge Rzx) \quad \equiv \neg\exists x \neg\exists yz.(Rxy \wedge Ryz \wedge Rzx)$

- No node lies on a 4-cycle (in <u>UNFO</u> but not in GF or FO^2)

  $\neg\exists xyzu.(Rxy \wedge Ryz \wedge Rzu \wedge Rux)$

- Ternary relation R is contained in S (in <u>GF</u> but not in UNFO or FO^2)

  $\forall xyz(Rxyz \rightarrow Sxyz) \quad \equiv \neg\exists xyz(Rxyz \wedge \neg Sxyz)$

- R is the total relation (in <u>FO^2</u> but not in GF or UNFO)

  $\forall xy.Rxy \quad \equiv \neg\exists xy \neg Rxy$

- UNFO and UNFP generalizes several existing logics:
  - Modal logic, modal mu-calculus, various description logics,
  - Unions of conjunctive queries, monadic Datalog,
  - CTL*(X), Core XPath

Conjunctive Queries

- Conjunctive query (CQ): $\exists \mathbf{x}(\phi_1 \wedge \ldots \wedge \phi_n)$
- Union of conjunctive queries (UCQ): a disjunction of CQs
- Conjunctive Queries play a fundamental role in DB theory.
- Decidable (NP-complete) for entailment

World of
modal logic

World of
database theory

UNFO

Basic Modal Language

(Unions of)
Conjunctive queries

UNFP

Modal mu-calculus

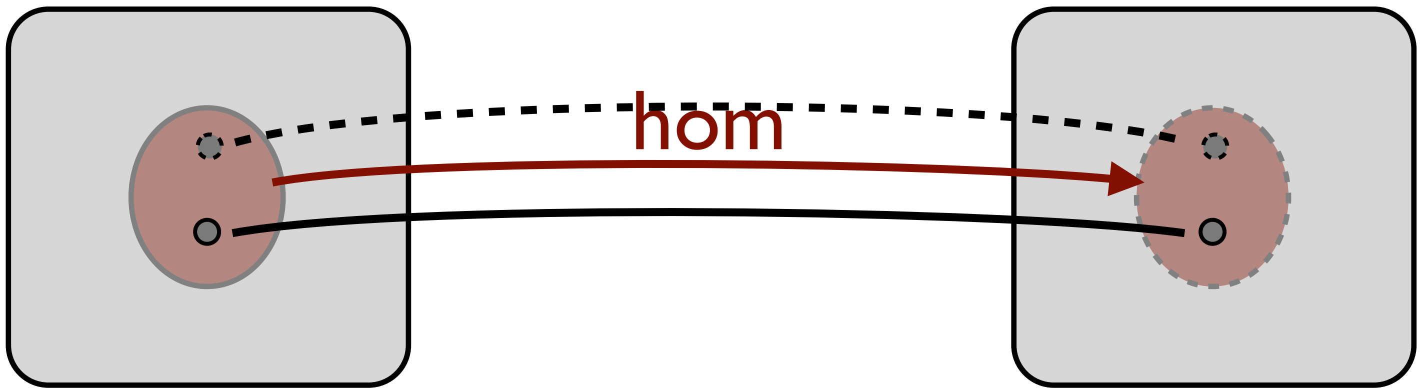
Monadic datalog

UN-bisimulations

Bisimulations

Homomorphisms

UN-bisimulation game



- The UN-bisimulation game:
  - Positions: pairs (a,b)
  - Moves:
    - Spoiler picks a finite set X in one of the structures.
    - Duplicator responds with a partial homomorphism h from X to the other structure (s.t. h(a)=b if a is in X).
    - Spoiler picks a pair (c,d) in h.

Model theory (i)

- **Thm**: UNFP is invariant for UN-bisimulations.
- **Thm**: UNFO is UN-bisimulation-invariant fragment of FO.

Model theory (ii)

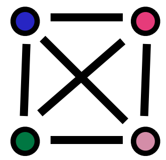
- **Thm**: UNFO and UNFP have the tree-like model property: every satisfiable formula ϕ is satisfied in a structure of tree-width at most $|\phi|$.

- **Thm**: UNFO has the finite model property: every satisfiable formula is satisfied in a finite structure.

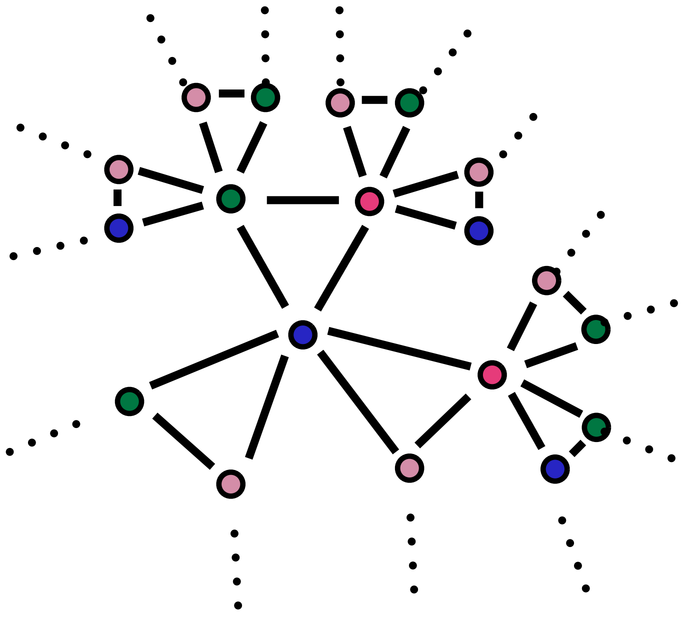
  (Finite model property fails for UNFP as witnessed by $\forall x \exists y (x<y) \wedge \forall x [\mathrm{LFP}_{X,y}\ \forall z (z<y \rightarrow Xz)](x)$)

- **Corollary**:
  - UNFP-satisfiability is decidable;
  - UNFO-satisfiability on finite structures is decidable.

UN-bisimilar tree-like unraveling



A structure



its UN-unraveling of width 3
(has treewidth 3)

Model theory (iii)

- **Thm**: UNFO has Craig interpolation.

  If $\phi(\mathbf{P},\mathbf{Q}) \models \psi(\mathbf{Q},\mathbf{R})$ then there is a formula $\chi(\mathbf{Q})$ such that $\phi(\mathbf{P},\mathbf{Q}) \models \chi(\mathbf{Q})$ and $\chi(\mathbf{Q}) \models \psi(\mathbf{Q},\mathbf{R})$

Complexity (i)

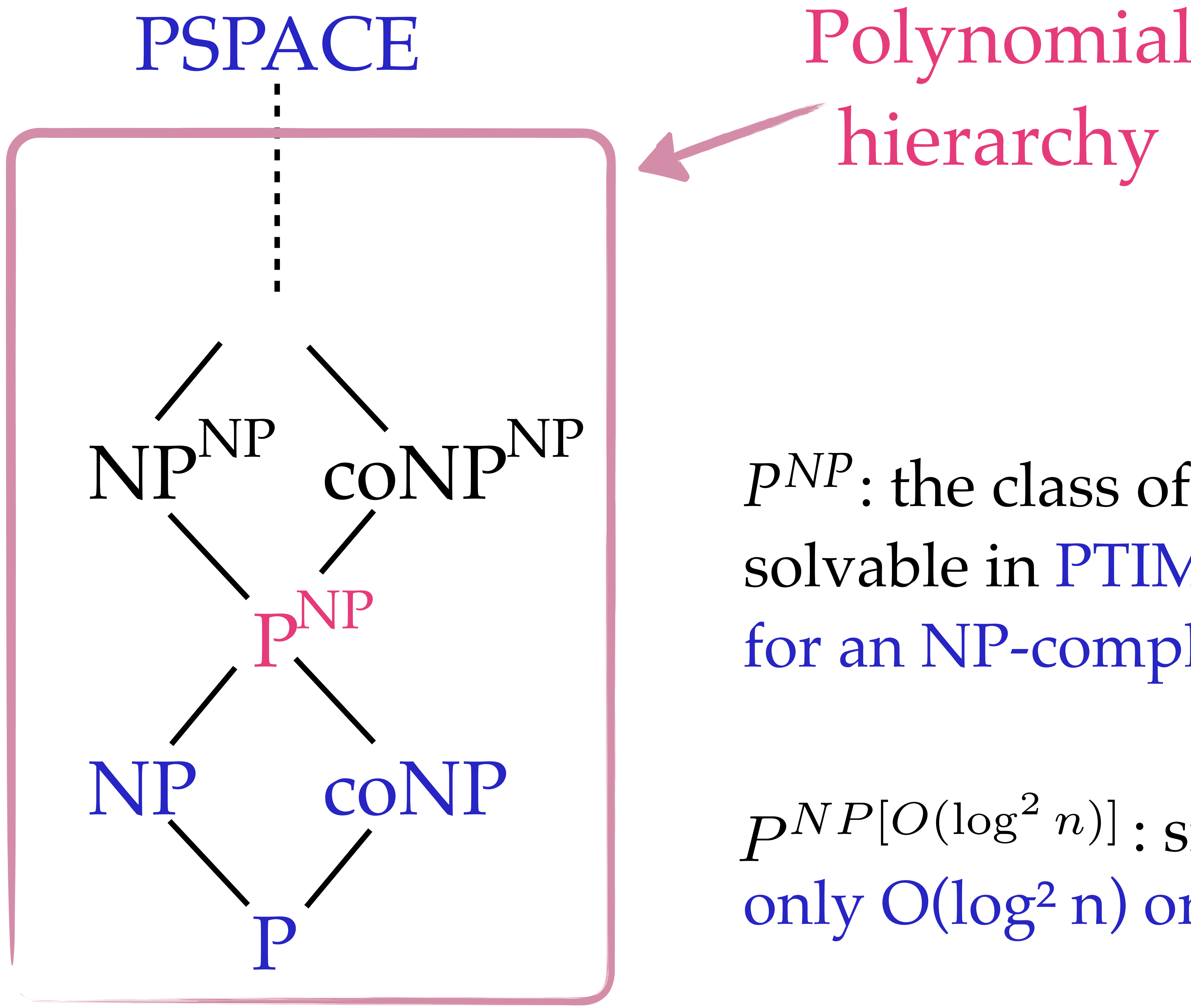
- **Thm**: Satisfiability, validity and entailment are 2ExpTime-complete for UNFO.
  - (Recall that UNFO has the Finite Model Property)

- **Thm:** Satisfiability, validity and entailment are 2ExpTime-complete for UNFP.
  - On arbitrary structures as well as finite structures.

(By a reduction to (finite) satisfiability for the two-way mu-calculus, cf. Bojanczyk '02)

Complexity (ii)

- Complexity of UNFO model checking?
  - In PSPACE (from full FO logic)
  - NP-hard (from Conjunctive Queries)
  - Complete for some class in the polynomial hierarchy that is called $P^{NP[O(\log^2 n)]}$



P^{NP}: the class of problems solvable in PTIME with an oracle for an NP-complete problem.

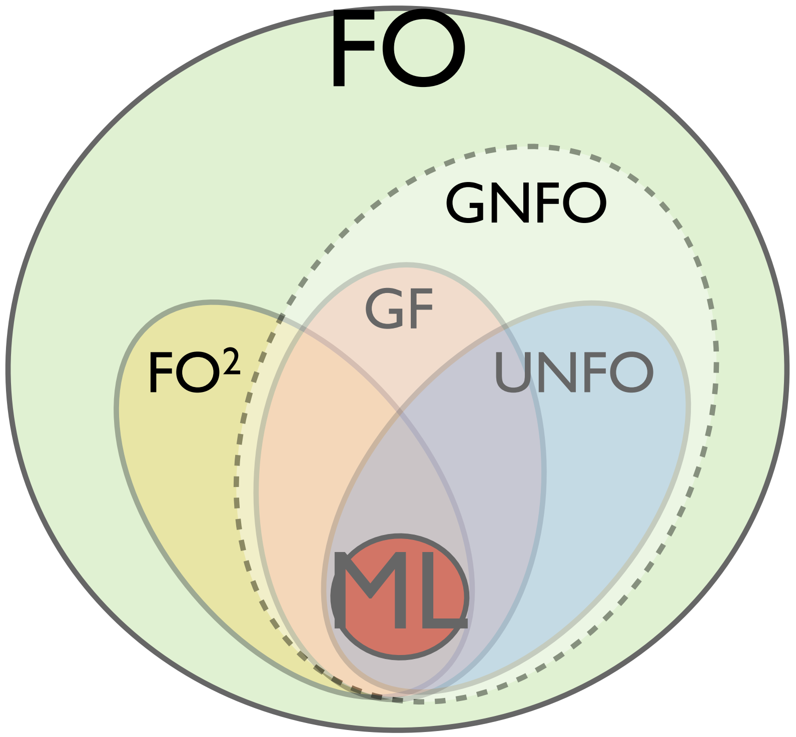
$P^{NP[O(\log^2 n)]}$: similar except with only O(log² n) oracle calls allowed.

Complexity (ii)

- Complexity of UNFO model checking?
  - In PSPACE (from full FO logic)
  - NP-hard (from Conjunctive Queries)
  - Complete for $P^{NP[O(\log^2 n)]}$
  - The temporal logic CTL*(X), for which $P^{NP[O(\log^2 n)]}$ completeness was known (Schnoebelen 2003), embeds into UNFO. Hence, lowerbound is for free.
- Complexity of UNFP model checking?
  - In $NP^{NP} \cap coNP^{NP}$ and P^{NP}-hard

Unary Negation vs Guardedness

- What do UNFO & UNFP have that GF & GFP do not have?
  - Unrestricted existential quantification, and therefore, includes Unions of Conjunctive Queries / monadic Datalog.
  - UNFO has Craig interpolation (which fails for GF)
- A common generalization: guarded negation [Barany-tC-Segoufin '11].
  - All results for unary negation have been lifted to guarded negation.



Guarded Negation

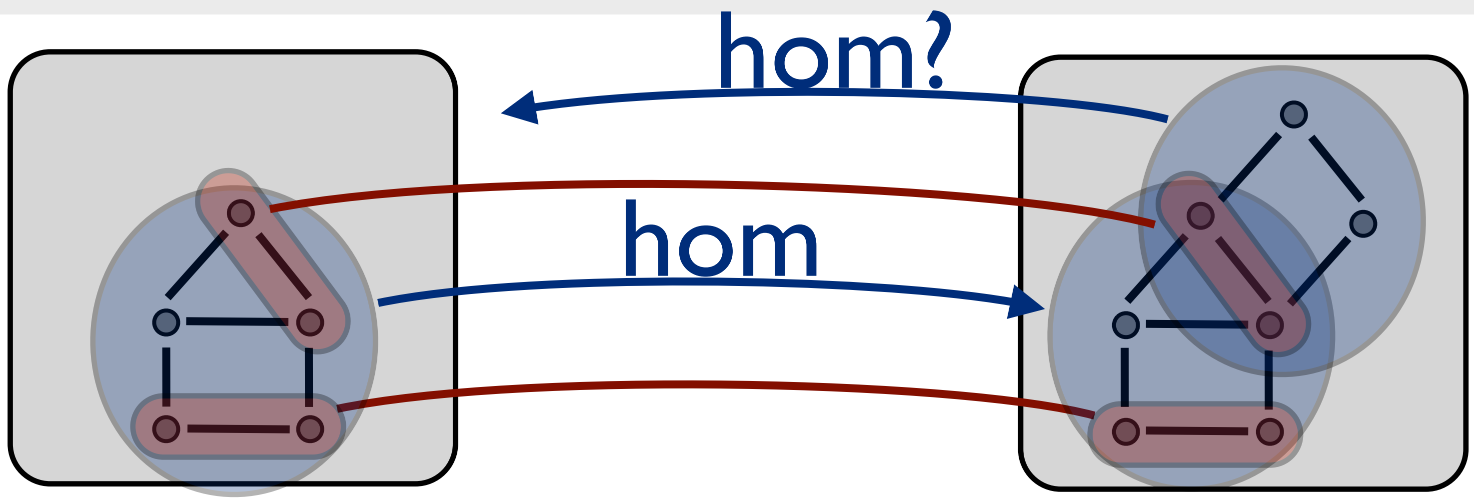
- Guarded Fragment (GF):
  - $\phi ::= R(\mathbf{x}) \mid x = y \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi \mid \exists\mathbf{x}\, G(\mathbf{xyz}) \wedge \phi(\mathbf{xy}) \mid \exists x\, \phi(x)$
  - Unrestricted use of negation; restricted use of quantification.
- Guarded Negation FO (GNFO):
  - $\phi ::= R(\mathbf{x}) \mid x=y \mid \exists x\phi \mid \phi \vee \phi \mid \phi \wedge \phi \mid \neg\phi(x) \mid G(\mathbf{xy}) \wedge \neg\phi(\mathbf{x})$
  - Restricted use of negation; unrestricted use of existential quantification.
- Fixed-point Extension (GNFP):
  - $\phi ::= \ldots \mid [\mathrm{LFP}_{Z,\mathbf{z}}\, \mathrm{guarded}(\mathbf{z}) \wedge \phi(\mathbf{Y}, Z, \mathbf{z})](\mathbf{x})$ (where Z is positive in ϕ)
- **Fact**: Every GF / GFP sentence is equivalent to a GNFO / GNFP sentence.

Normal form

- GN-Normal form for GNFO formulas: $q[\phi_1/U_1, \ldots, \phi_n/U_n]$ (where q is a UCQ with relations $U_1, \ldots, U_n$)
  - $\phi ::= R(\mathbf{x}) \mid x{=}y \mid \cancel{\exists x\phi \mid \phi\vee\phi \mid \phi\wedge\phi} \mid \neg\phi(x) \mid G(\mathbf{xy})\wedge\neg\phi(\mathbf{x})$
  - I.e., ϕ is built up from atoms using (i) UCQs, and (ii) guarded negation.
- GN-Normal form for GNFP formulas:
  - $\phi ::= R(\mathbf{x}) \mid x{=}y \mid q[\phi_1/U_1, \ldots, \phi_n/U_n] \mid \neg\phi(x) \mid G(\mathbf{xy})\wedge\neg\phi(\mathbf{x})$

    $\mid \mathrm{LFP}_{Z,\mathbf{z}}[\, \mathrm{guarded}_\sigma(\mathbf{z}) \wedge \phi(\mathbf{Y},Z,\mathbf{z}) \,](\mathbf{x})$
  - I.e., ϕ is built up from atoms using (i) UCQs, (ii) guarded negation, and (iii) guarded LFPs.
- Every GNFO / GNFP formula is equivalent to one in GN-normal form.

GN-Bisimulation Game



- The GN-bisimulation game:
  - Positions: pairs of guarded sets (**a**,**b**)
  - Moves:
    - Spoiler picks a finite set X in one of the structures.
    - Duplicator responds with a partial homomorphism h from X to the other structure, s.t. h(**a**)=**b**.
    - Spoiler picks guarded subsets (**c**,**d**) in h.

Querying the Guarded Fragment

- Barany-Gottlob-Otto 2010 (“Querying the guarded fragment”):
  - The following is 2ExpTime-complete and finitely controllable:

    Given a GF-sentence ϕ and a (Boolean) UCQ q, test if $\phi \models q$.
- GNFO is a common generalization of GF and UCQs.
  - The above question is equivalent to the (un)satisfiability of $\phi \land \neg q$.
  - Conversely, GNFO satisfiability reduces to querying the guarded fragment.
    - Replace all UCQs in the formula by fresh predicates, and “axiomatize” them (a la Scott normal form) using GF sentences and negated CQs.
  - We show GNFP satisfiability is 2ExpTime-complete using the techniques from [Barany-Gottlob-Otto 2010] as well as [Barany-Bojanczyk 2011].

- All results for UNFO and UNFP go through also for GNFO and GNFP.
  - (Craig interpolation for GNFO was still open until very recently, but solved with Vince Barany and Michael Benedikt, not yet published)

- Remainder of the talk:
  1. More about Craig interpolation
  2. More about tree width
  3. About the proof of decidability of finite satisfiability for GNFP.

Failure of Craig Interpolation in GF

- Craig interpolation fails for GF, although a weak form of interpolation holds [Hoogland and Marx '00]
- Let $\phi(x_1) = \exists x_2,\ldots,x_n\ (\ G(x_1,\ldots,x_n) \wedge \text{DIRECTED-R-CYCLE}(x_1,\ldots,x_n)\ )$
- Let $\psi(x_1) = P_0(x_1) \wedge \bigwedge_i \forall xy(P_i(x) \wedge R(x,y) \rightarrow P_{i+1}(y)) \rightarrow P_n(x_1)$
- $\phi(x_1)$ implies $\psi(x_1)$, and every interpolant is provably equivalent to $\exists x_2,\ldots,x_n\ \text{DIRECTED-R-CYLE}(x_1,\ldots,x_n)$
- The latter formula is not invariant for guarded bisimulations (n>2).
- **Conclusion 1**: GF lacks Craig interpolation
- **Conclusion 2**: Any extension of GF with Craig interpolation must contain (a formula equiv. to) $\exists x_2,\ldots,x_n\ \text{DIRECTED-R-CYLE}(x_1,\ldots,x_n)$

Tree width

Trees

Grids

Trees

- For any relational structure M define the Gaifman graph of M:
  - The nodes are the elements of M
  - An edge represents that two elements co-occur in a tuple of a relation.
- A tree structure is a structure whose Gaifman graph is a tree.
- Satisfiability of FO formulas on tree structures is decidable
- In fact, this holds even
  - for Monadic Second-Order logic (MSO) ...
  - ... on structures of “bounded tree width” (Rabin; Seese; Courcelle)

Beyond MSO

- Monadic Second-Order Logic (MSO):
  - Extension of FO with quantification over sets.
- Guarded Second-Order Logic (GSO):
  - Further extension of MSO with quantification over guarded relations (i.e,. relations containing only guarded tuples)
- “Counting quantifiers” and “Finiteness quantifiers”
  - “There are p mod q many …” and “There are finitely many …”

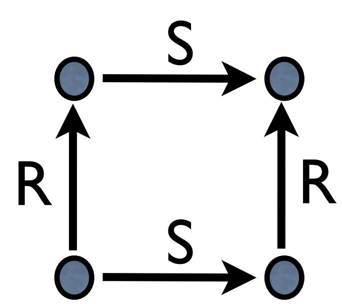
Tree width

- **Tree width**: a measure of tree-likeness.
  - trees and forests have tree width 1,
  - cycles have tree width 2,
  - an n x m grid has tree width min{n,m}

- There are several equivalent ways to define tree width.
  - Via tree decompositions
  - Via cops and Robber games,
  - ...

Positive existential descriptions

- The positive existential description of a finite structure M:



$$\exists x_1 x_2 x_3 x_4 (Rx_1x_2 \wedge Sx_2x_4 \wedge Sx_1x_3 \wedge Rx_3x_4)$$

- The following are equivalent:
  - A structure N satisfies the positive existential description of M
  - There is a homomorphism $h : M \rightarrow N$.

Unusual Definition of Tree-Width

- Fact about finite trees:
  - The positive existential description can be rewritten to an $FO^{\exists\wedge+}$ formula with only 2 variables (reusing is allowed).
- (Unusual) definition of tree width for finite structures
  - A finite structure M has tree width *k* if its positive existential description is equivalent to a $FO^{\exists\wedge+}$ with at most *k+1* variables, *even after each node is labeled with a distinct additional unary predicate.* (Kolaitis & Vardi '00)
- The tree width of an infinite structure is the supremum of the tree width of its finite substructures (can be shown by a compactness argument)

Grids versus bounded tree width

- Large grids have a high tree width.
- Conversely, graphs of high tree width contain large grids as a graph minor (Robertson & Seymour's Excluded Grid theorem).
- Thus, bounding the tree width ≈ disallowing large grids.

The Border of Decidability

- The following is decidable (Rabin; Seese; Courcelle):
  - Given a $\phi \in$ GSO and k>0, does ϕ have a model of tree width $\leq k$.
  - This holds even for GSO + Counting + Finiteness
- Conversely, if GSO satisfiability is decidable on a class of structures C, then the structures in C have bounded tree width. [Seese 1991]
- In other words, "if it's decidable, it's because of bounded tree width"
- Complexity: non-elementary (= time needed to solve the problem cannot be bounded by a finite stack of exponentials)

Decidability of GNFP Satisfiability

- **Thm**: (Finite) satisfiability of GNFP is 2ExpTime complete.
- Main ingredients of the proof:
  - Treeifications of cyclic conjunctive queries,
  - Locally acyclic covers for the guarded fragment ([Barany-Gottlob-Otto 2010])
  - A reduction from GNFP to GFP.
  - Decidability of finite satisfiability for GFP ([Barany-Bojanczyk 2011])

Locally Acyclic Covers

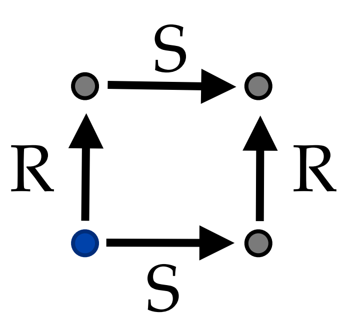
- Tree-Unraveling of a Kripke structure:
  - a simple construction that removes cycles at the cost of making the structure infinite.
  - the tree-unraveling of a structure is bisimilar to the original, hence modal formulas are preserved.
- Locally Acyclic Covers (Otto 2004):
  - a more involved “finitary unraveling” construction that make a structure locally acyclic (no cycles of length less than N) .
  - Useful to prove finite model theory versions of Van Benthem bisimulation theorems (cf. Dawar and Otto 2009)

Locally Acyclic Covers for the Guarded Fragment

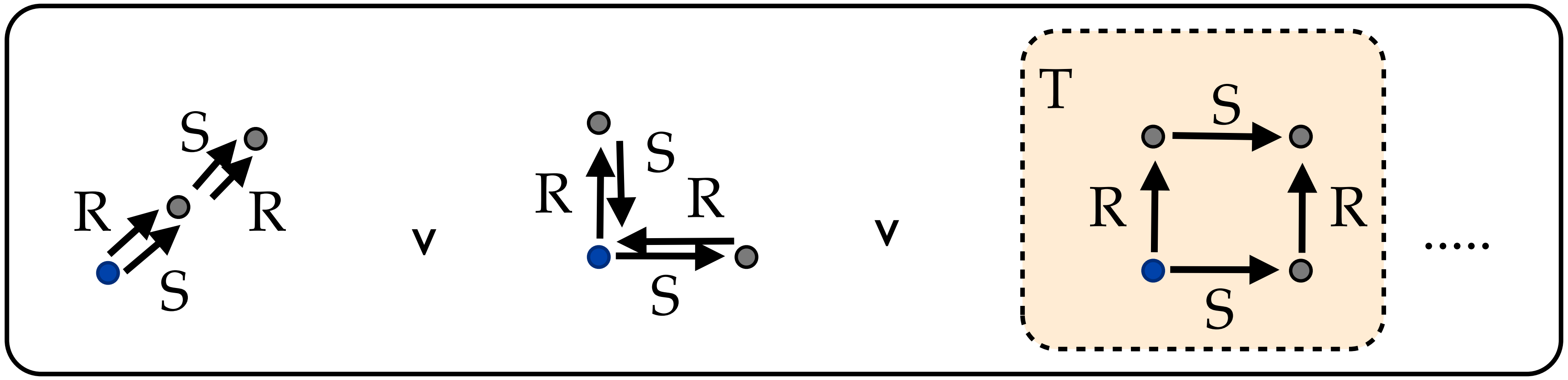
- Barany, Gottlob, Otto (2009): an analogous construction for the guarded fragment.
  - Every finite structure is GF-bisimilar to a finite structure whose hypergraph is “locally acyclic” (suitably defined).

Treeifying Conjunctive Queries

- **Treeification Lemma**: for every conjunctive query q there is a GF-formula that describes all ways in which q may be realized in a locally acyclic structure.



A (cyclic) CQ



Its treeification

- This provides a way of reducing GNFO to GF, and GNFP to GFP.

- Thank you!