Constants and Consequence Relations

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Joint work with Denis Bonnay (Université Paris Ouest Nanterre),

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  - allows a distinction between logical and analytic inference.

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Hence: Phil is a bachelor.

(2) Every bachelor is mortal.
Henry is not a bachelor.
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Another motivation is a general interest in consequence relations.

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$\vdash$ is compact if $\Gamma \vdash \phi$ implies $\Gamma' \vdash \phi$ for some finite $\Gamma' \subseteq \Gamma$.

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(c) There is a standard interpretation $I_L \in INT_L$.

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Let

$CONS_L$

be the set of consequence relations in L that preserve truth in the standard interpretation I_L.

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So $\Rightarrow_{\_}$ is a monotone map from $\mathcal{P}(Symb_L)$ to $CONS_L$.

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$C_{\_}$ can be shown to extract exactly the expected logical constants from the consequence relations of standard logics (be they Bolzano-Tarski consequence relations or not).

Moreover, $C_{\_}$ is (under certain assumptions) an inverse to $\Rightarrow_{\_}$ in the sense of giving a (monotone) Galois connection.

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Fact

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Monotone GCs generalize order isomorphisms as follows: Let the left kernel of the GC be $g(f(A))$, and let the right kernel be $f(g(B))$. Then the kernels are isomorphic (so, restricted to the kernels, $g = f^{-1}$).

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Theorem (Bonnay and W-l, 2012)

(a) $(C_{\_}, \Rightarrow_{\_})$ *is a monotone GC between* $(BTCONS_L^c, \subseteq)$ *and* $(\mathcal{P}(Symb_L), \subseteq)$.

(b) *The left kernel is $BTCONS_L^c$ itself. The right kernel is the set of minimal sets of symbols: X is minimal iff no smaller set generates the same consequence relation.*

A Galois connection, cont.

This theorem is a special case of a more general result, holding not only for Tarskian interpreted languages, but languages specified purely syntactically.

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This is effected by looking not just at L but (directed) families $\mathcal{L}$ of expansions of L, and relativizing consequence, extraction, etc. to $\mathcal{L}$.

A limitation of $C_{\_}$

Compare:

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Hence: Phil is a bachelor.

(2) Phil is good-looking *and* he is a bachelor
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This would require another method of extracting constants.

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Recall the notion of a Tarskian interpreted language L, with a notion of truth in L-interpretations, and a standard interpretation I_L.

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This is a (familiar) variant of an observation going back to Lawvere (1969) (cf. $Mod(\Psi)$ vs. $Th(K)$).

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We consider two ways of extracting constants from classes of interpretations.

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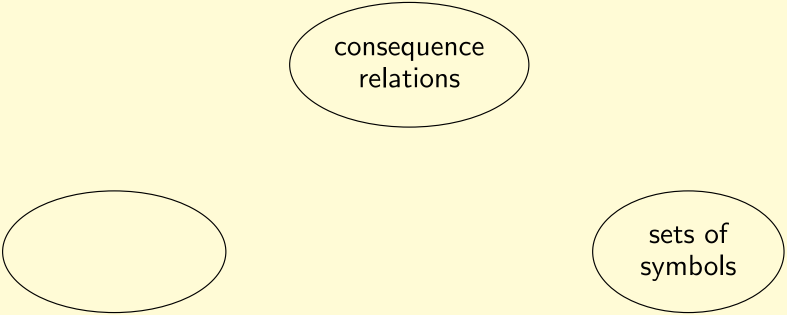
Now we have three ordered structures of interest.

three classes of interest (given L)



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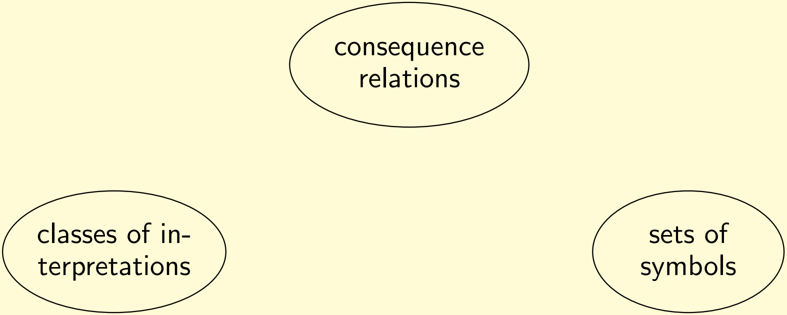
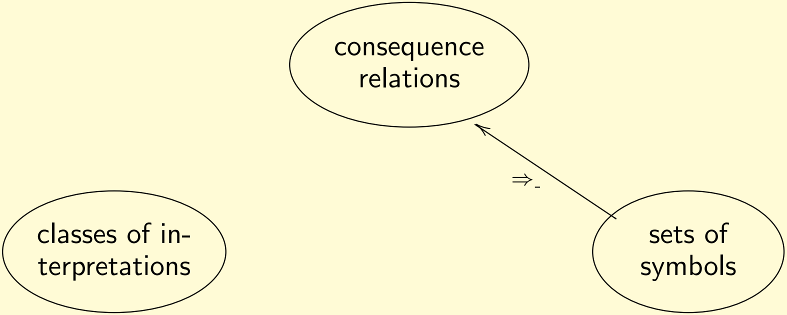
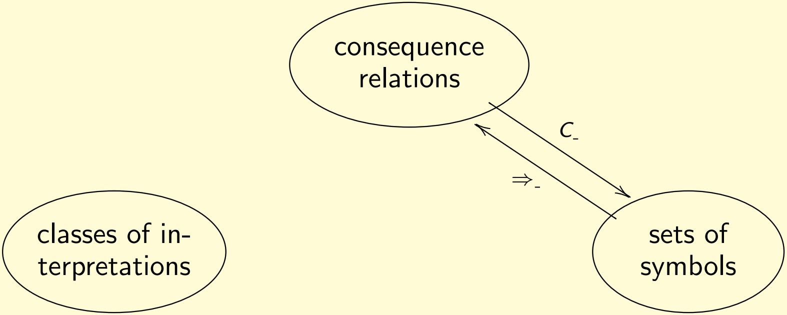


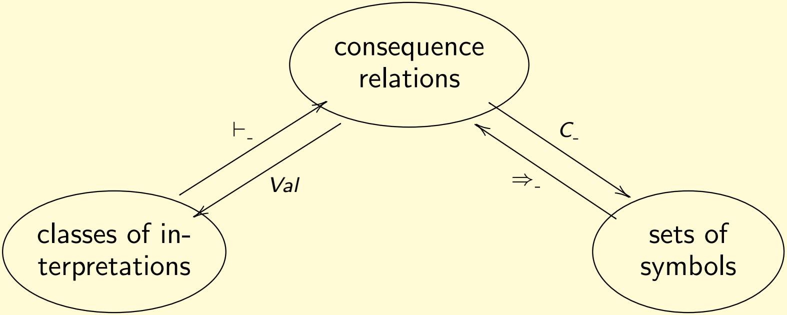
the Bolzano-Tarski map



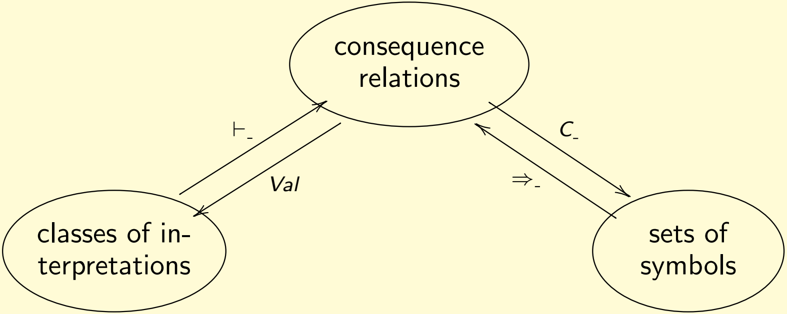
the syntactic extraction map $C_{\_}$



two more familiar maps



no composed Galois connection



These two Galois connections do not compose to a third (antitone) one, since the right one is restricted to consequence relations of the form $\Rightarrow_Y$.

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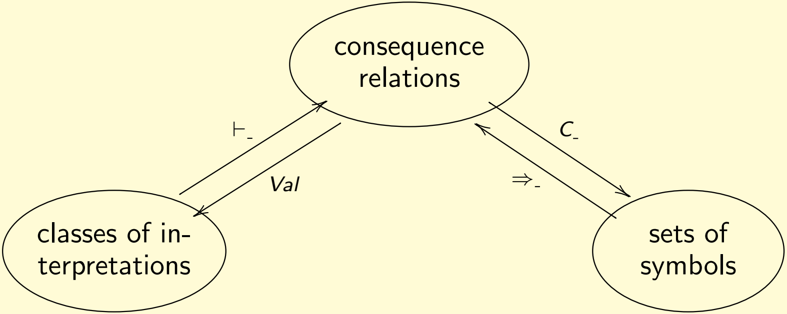

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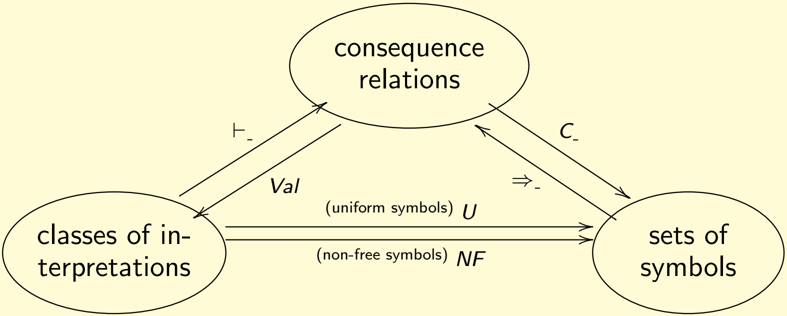
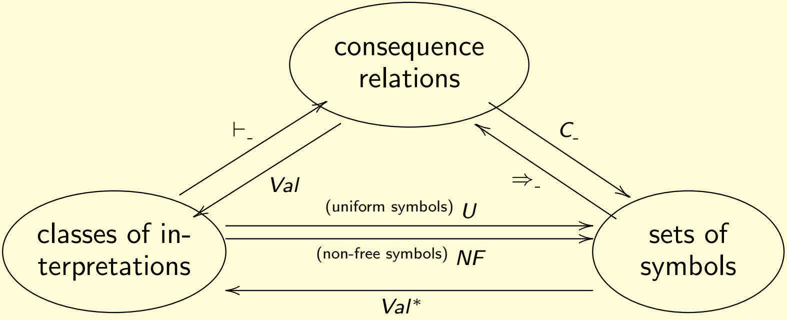
But we have other extraction methods from classes of interpretations.

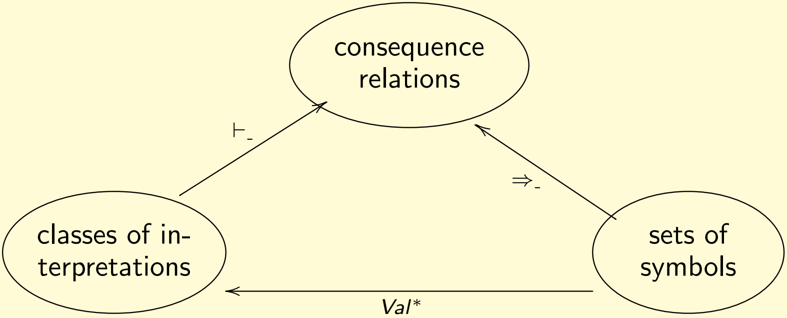
extracting symbols from classes of models



and in the other direction



a commutative triangle



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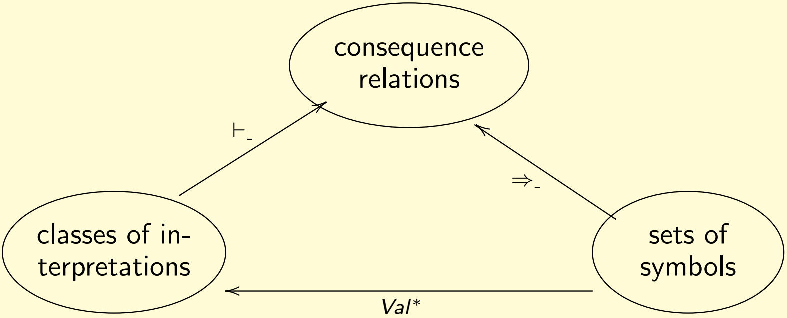
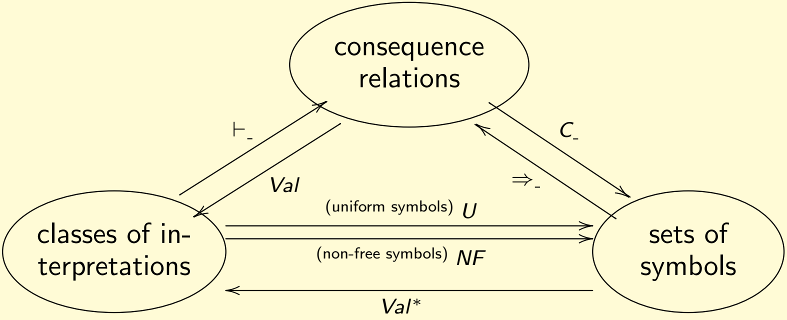


$$\vdash_{Val^*(X)} \;=\; \Rightarrow_X$$

the full picture



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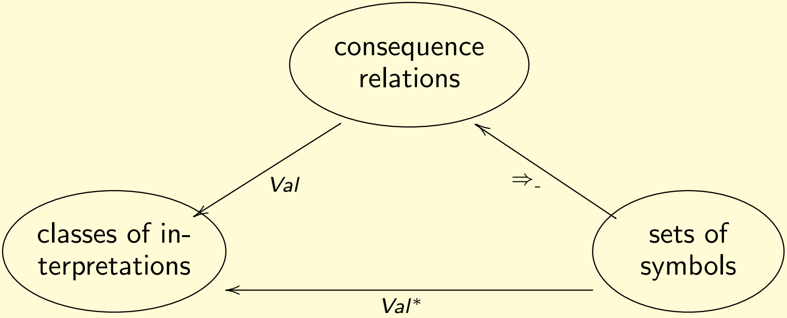

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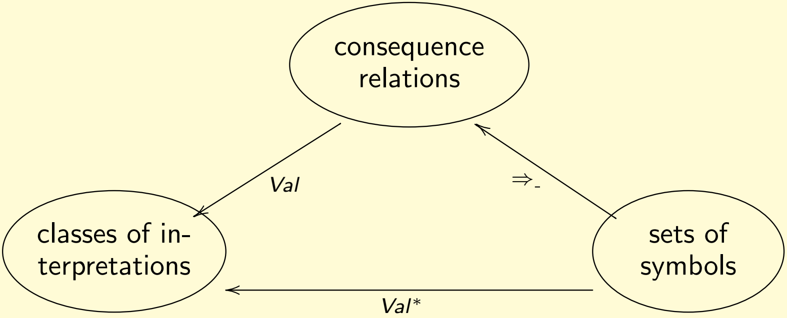

$$Val^*(X) \ = \ Val(\Rightarrow_X)\,?$$

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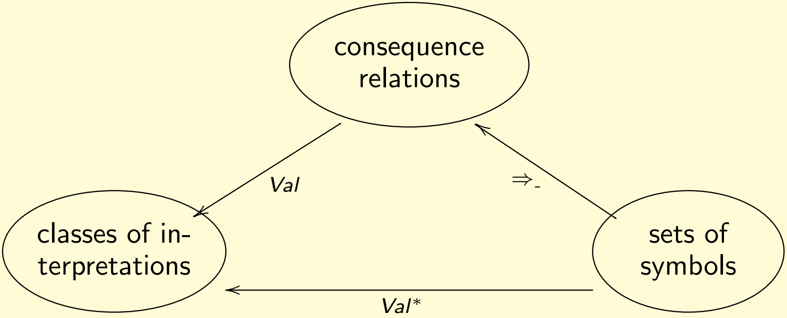
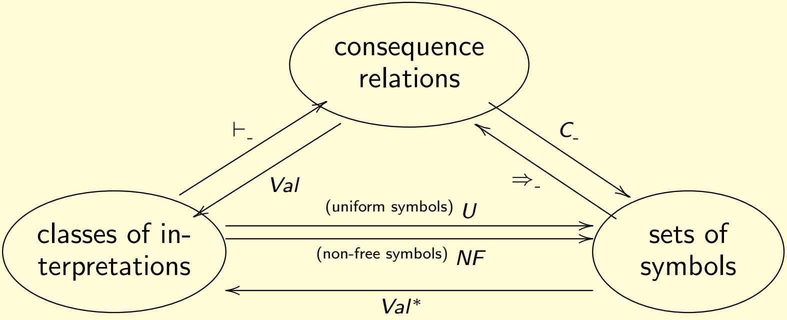


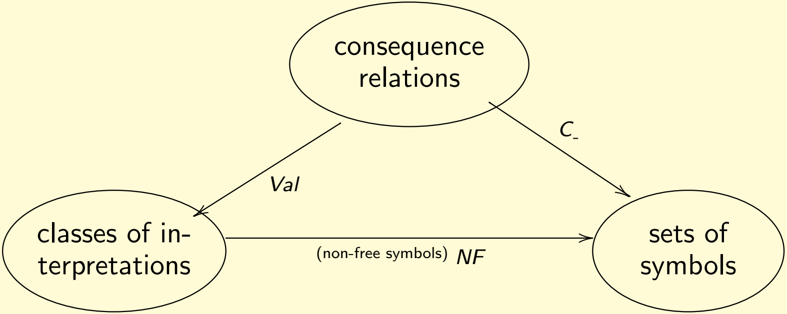
$Val^*(X) = Val(\Rightarrow_X)$? No, only $Val^*(X) \subseteq Val(\Rightarrow_X)$

($Val(\Rightarrow_X)$ is usually much bigger: 'non-standard models' exist)

the full picture again



What about *NF* and *C*_?



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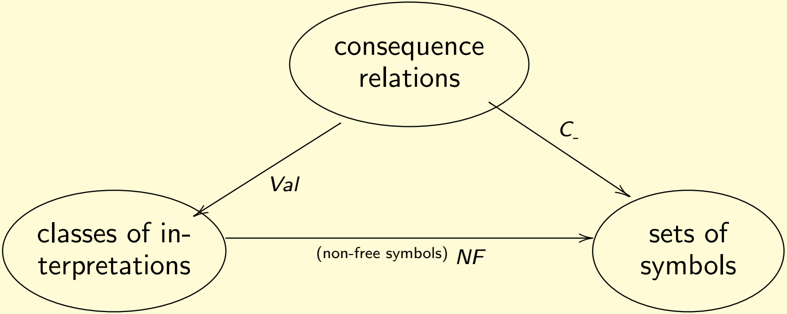

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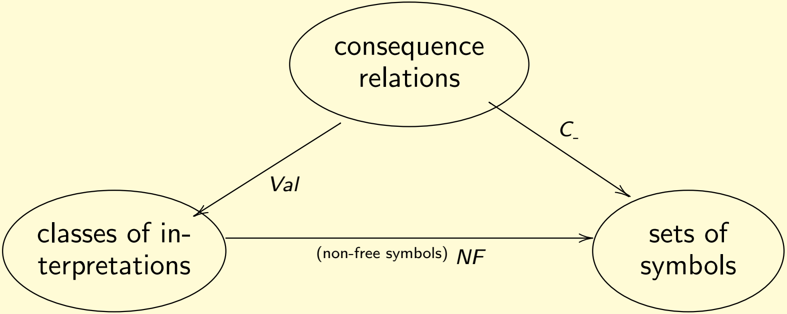

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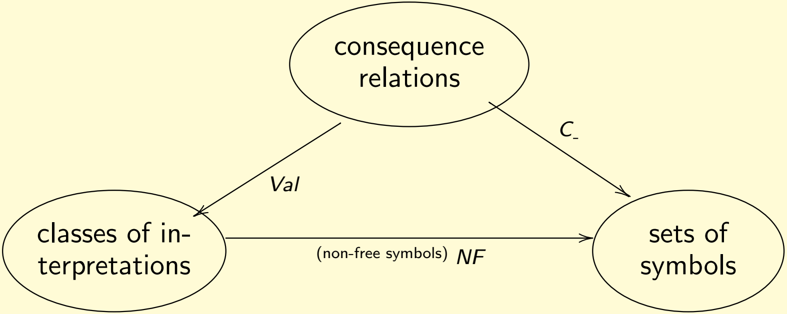

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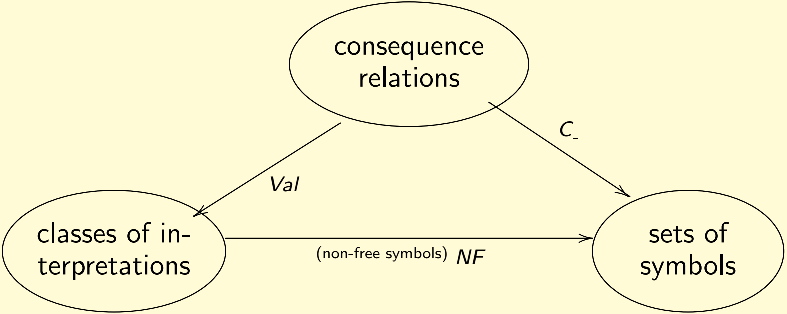

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NF and $C_{\text{-}}$, cont.

$C_{\text{-}}$, but not NF, depends on the syntax. In order to compare NF and $C_{\text{-}}$, we need to ease these limitations.

NF and $C_{\_}$, cont.

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We introduce a family of expansions $\mathcal{L}$ (adding new symbols, arbitrarily interpretable) for a base language L, with corresponding families of consequence relations $\vdash_{K,\mathcal{L}} = \{\vdash_{K,L'}\}_{L' \in \mathcal{L}}$, and we adapt the definitions of Val, NF, and $C_{\_}$ accordingly.

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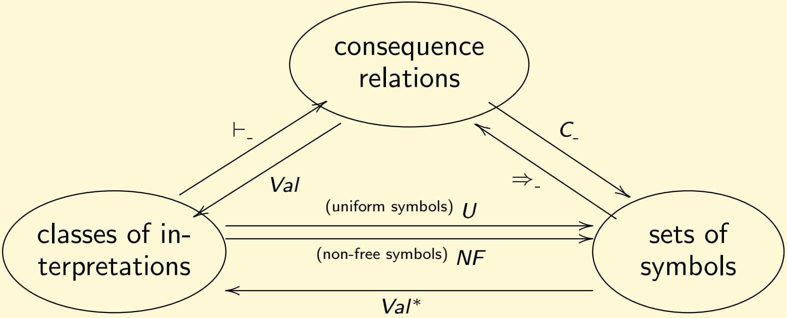
So NF is, after all, a semantic version of $C_\_$.

the full picture again



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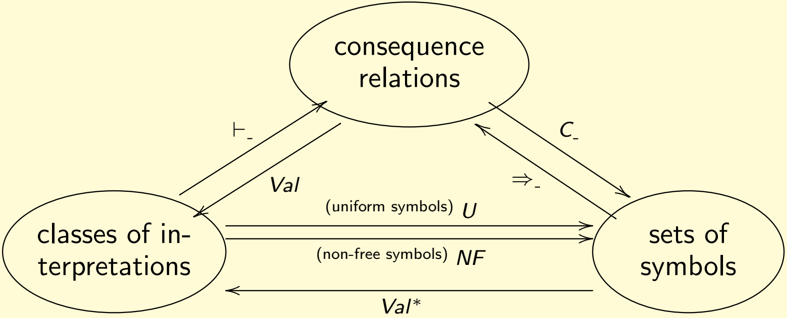
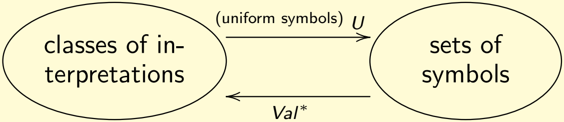


We do have a Galois connection also along the base of the triangle:

A Galois connection for U



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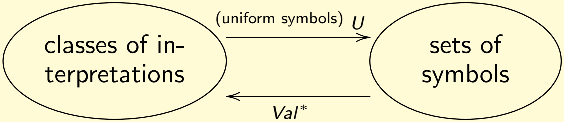

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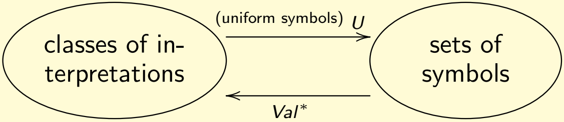

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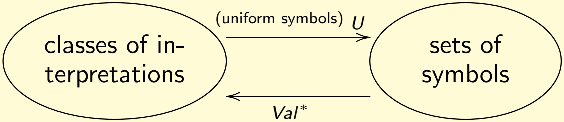

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Observe that

$$U(K) = \{u\colon \text{for all } I \in K, I =_{\{u\}} I_L\}$$

A Galois connection for U

$$\begin{tikzcd}
\text{classes of interpretations} \arrow[r, shift left, "\text{(uniform symbols) } U"] & \text{sets of symbols} \arrow[l, shift left, "Val^*"]
\end{tikzcd}$$

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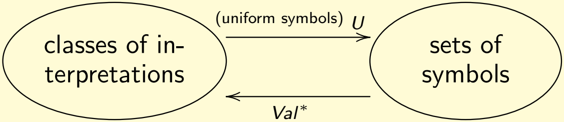

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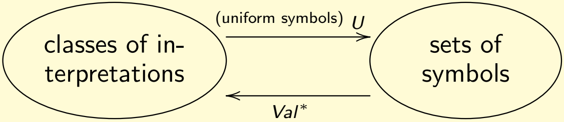

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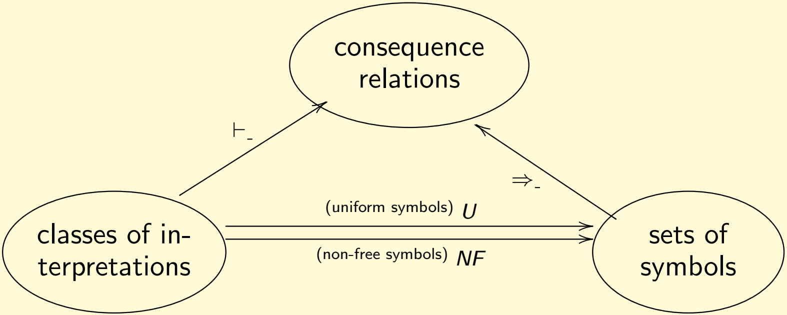
Nothing similar works for NF, which is neither monotone nor antitone.

U vs. NF, and how $\Rightarrow_{U(K)}$ and $\Rightarrow_{NF(K)}$ relate to $\vdash_K$



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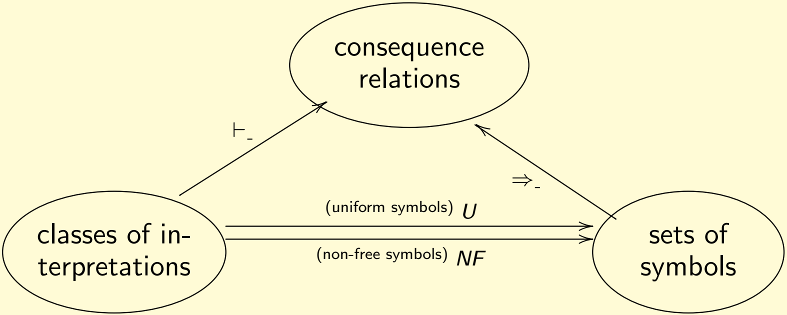

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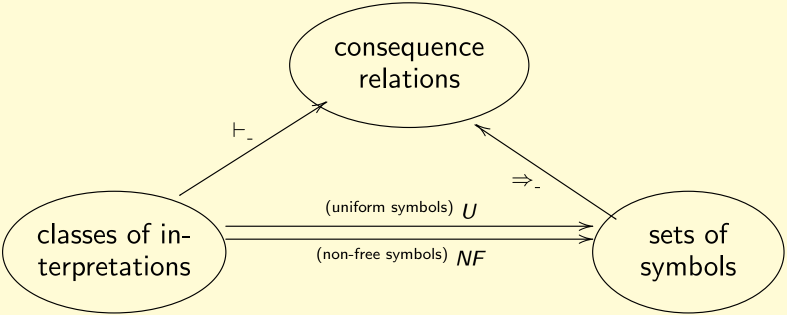

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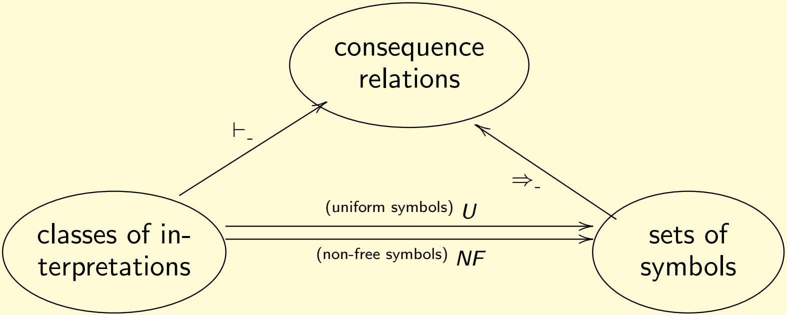

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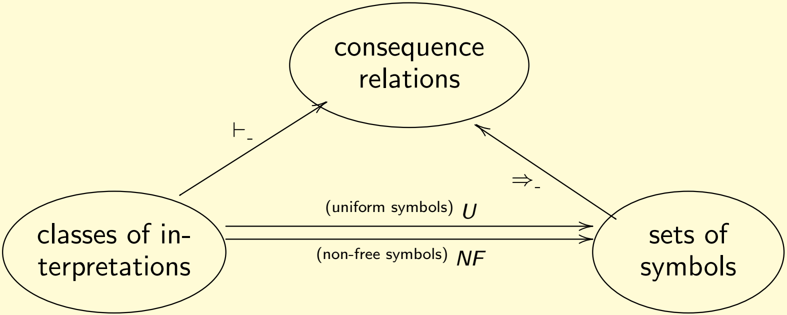

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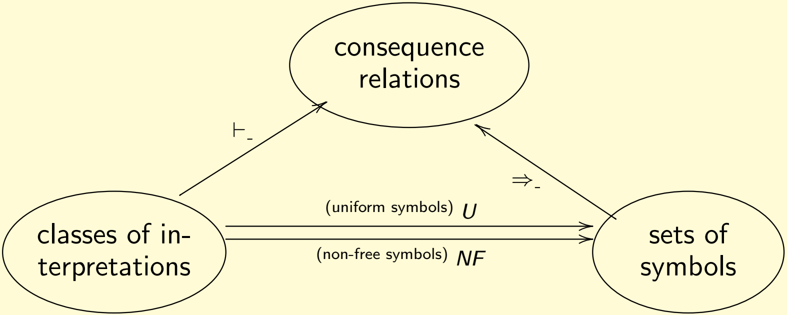

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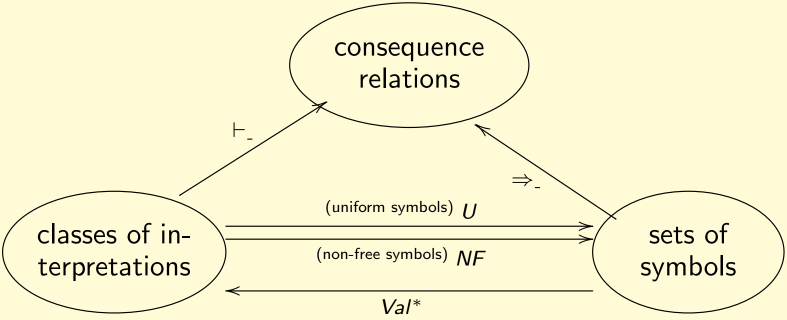

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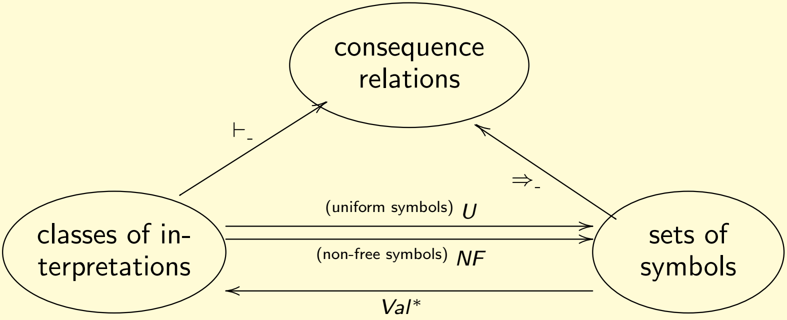

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(c) $\Rightarrow_{U(K)} \subseteq \vdash_K$ *and, if $Symb_L$ is finite,* $\vdash_K \subseteq \Rightarrow_{NF(K)}$.

Tarskian approximations of consequence relations

Since

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We can also show

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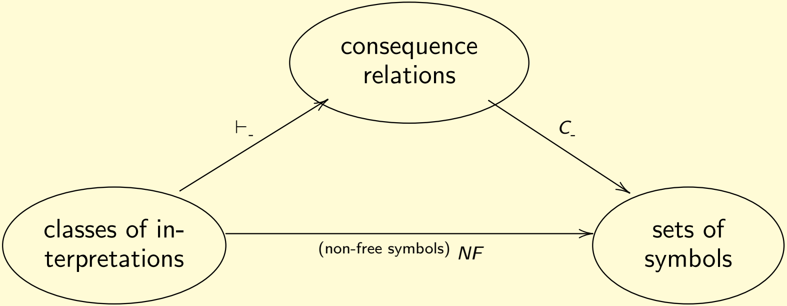
There are $\vdash_K$ and X such that $\vdash_K \subseteq \Rightarrow_X \subsetneq \Rightarrow_{NF(K)}$.

a final triangle: $NF(K)$ vs. $C_{\vdash_K}$



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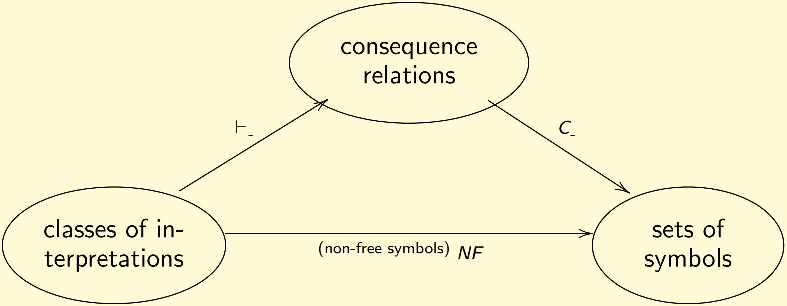

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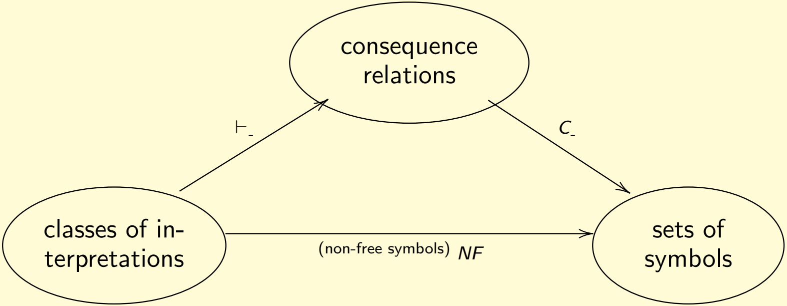

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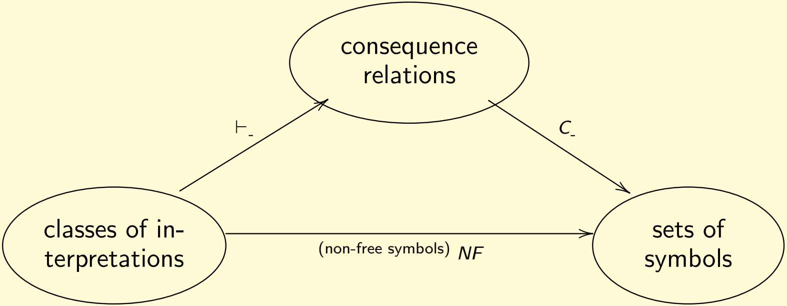

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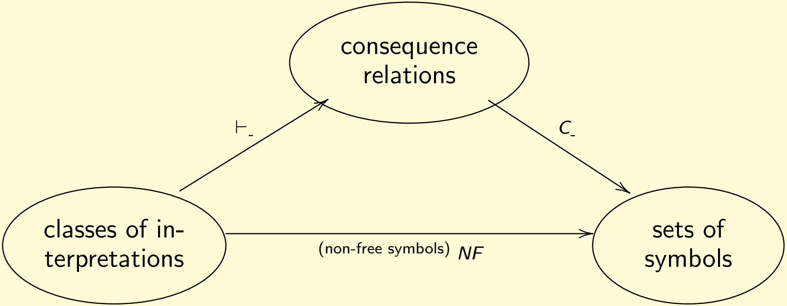

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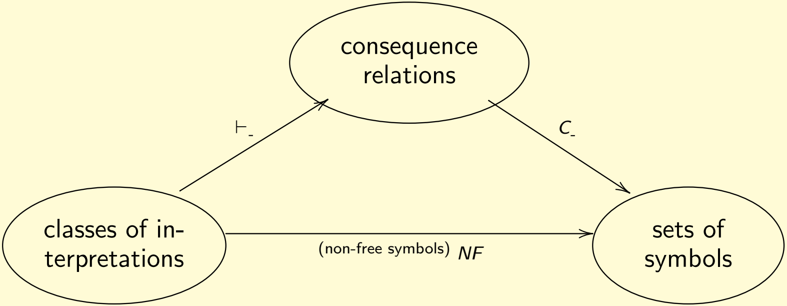

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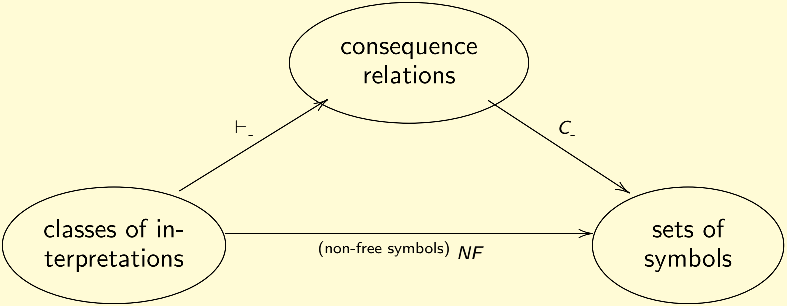

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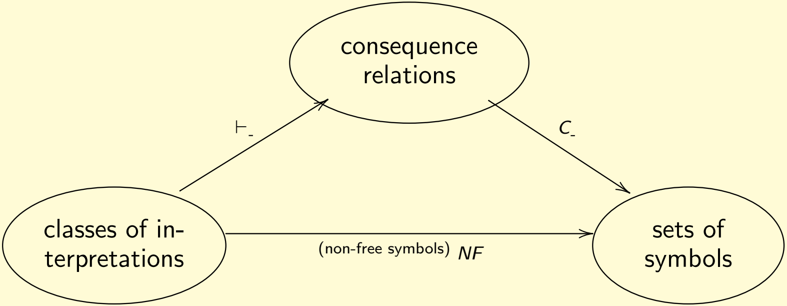

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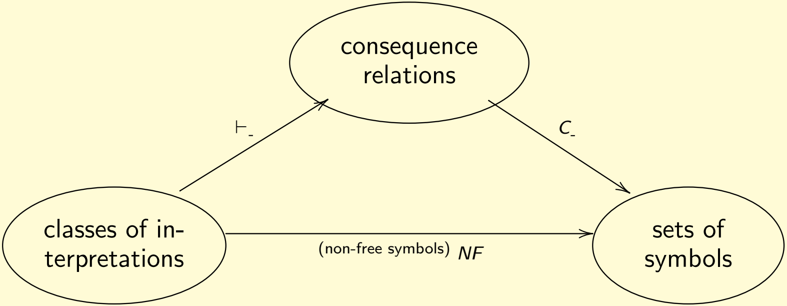

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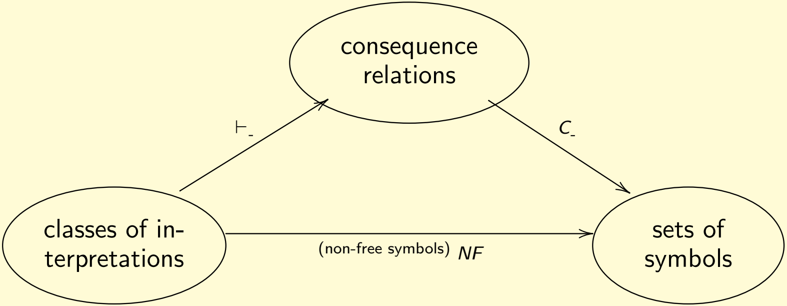

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U and analytic inferences

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(We saw that $C_{\_}$, and hence NF, cannot do that.)

U and analytic inferences, cont.

Let $\vdash$ be a pretheoretic notion of consequence, validating both (3) and (4):

(3) Phil is good-looking *and* he is a bachelor
Hence: Phil is a bachelor.

(4) Phil is good-looking *and* he is a *bachelor*
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Let us check this in a simple example.

U and analytic inferences, cont.

Let $Symb_L = \{\neg, \wedge, \vee, John, Phil, married, man, bachelor\}$.

U and analytic inferences, cont.

Let $Symb_L = \{\neg, \wedge, \vee, John, Phil, married, man, bachelor\}$. Standard interpretation I_L as expected; say, John and Phil are both bachelors.

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NB L-interpretations interpret all symbols, according to their categories.

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Let $\vdash$ be classical PL-consequence closed under the meaning postulates

(MP1) $bachelor(a) \vdash man(a) \wedge \neg married(a)$

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(5) $Phil \notin U(Val(\vdash))$

Proof.
Let I be like I_L, except that $I(Phil) = I_L(John)$. This disturbs no $\vdash$-consequences, so $I \in Val(\vdash)$. Thus, $Phil \notin U(Val(\vdash))$. $\square$

(6) $bachelor \notin U(Val(\vdash))$

Proof.
Let I be like I_L, except that $I(bachelor) = \emptyset$ and $I(married) = M$. MP1 and MP2 are still true, hence all other sequents in $\vdash$ as well. $\square$

U and analytic inferences, cont.

Let $Symb_L = \{\neg, \wedge, \vee, John, Phil, married, man, bachelor\}$. Standard interpretation I_L as expected; say, John and Phil are both bachelors.

NB L-interpretations interpret all symbols, according to their categories.

Let $\vdash$ be classical PL-consequence closed under the meaning postulates

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NB $Phil, bachelor$ belong to $C_\vdash$ and $NF(Val(\vdash))$!

U and analytic inferences, cont.

Recall: $Symb_L = \{\neg, \wedge, \vee, John, Phil, married, man, bachelor\}$.

(7) $\wedge \in U(Val(\vdash))$

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U and $\wedge$

How do the rules for $\wedge$ fix its interpretation?

φ	ψ	$\varphi \wedge \psi$
1	1	
1	0	
0	1	
0	0	

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$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}$$

φ	ψ	$\varphi \wedge \psi$
1	1	
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1	1	1
1	0	0
0	1	0
0	0	0

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φ	ψ	$\varphi \vee \psi$
1	1	
1	0	
0	1	
0	0	

U and $\vee$

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$$\frac{\varphi}{\varphi \vee \psi}$$

φ	ψ	$\varphi \vee \psi$
1	1	
1	0	
0	1	
0	0	

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φ	ψ	$\varphi \vee \psi$
1	1	1
1	0	1
0	1	
0	0	

U and $\vee$

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$$\frac{\varphi}{\varphi \vee \psi}$$

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φ	ψ	$\varphi \vee \psi$
1	1	1
1	0	1
0	1	
0	0	

U and $\vee$

How do the rules for $\vee$ fix its interpretation?

$$\frac{\varphi}{\varphi \vee \psi}$$

$$\frac{\psi}{\varphi \vee \psi}$$

φ	ψ	$\varphi \vee \psi$
1	1	1
1	0	1
0	1	1
0	0	

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How do the rules for $\vee$ fix its interpretation?

$$\frac{\varphi}{\varphi \vee \psi}$$

$$\frac{\psi}{\varphi \vee \psi}$$

$$\frac{\varphi \vee \psi \qquad \begin{matrix}[\varphi] \\ \vdots \\ \theta\end{matrix} \qquad \begin{matrix}[\psi] \\ \vdots \\ \theta\end{matrix}}{\theta}$$

φ	ψ	$\varphi \vee \psi$
1	1	1
1	0	1
0	1	1
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1	1	1
1	0	1
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Doing this for Tarskian interpreted languages in general is work in progress.

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But exactly how well U performs needs to be assessed further.

THANK YOU