Fast Growing Functions and Arithmetical Independence Results

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§1. Intro – A Mathematical Incompleteness

Are there any genuine *mathematical* examples of incompleteness?

Goodstein Sequences (1944) – Kirby and Paris (1982)

- Take any number a, for example $a = 16$.
- Write a in "complete base-2", thus $a = 2^{2^2}$.
- Subtract 1, so the base-2 representation is $a - 1 = 2^{2+1} + 2^2 + 2^1 + 1$.
- Increase the base by 1, to produce the next stage $a_1 = 3^{3+1} + 3^3 + 3^1 + 1 = 112$.
- Continue subtracting 1 and increasing the base: $a, a_1, a_2, a_3, \ldots$. Example: 16, 112, 1284, 18753, 326594, ...

Theorem (Part 1 – Goodstein, Part 2 – Kirby & Paris)

(1) Every Goodstein sequence eventually terminates in 0.
(2) But this is not provable in Peano Arithmetic (PA).

§1.1. The "Hardy" Functions

If, throughout any Cantor Normal Form $\alpha \prec \varepsilon_0$, we replace ω by a large enough n, then we obtain a "complete base-n" representation. Subtracting 1, and then putting the ω back, one gets a smaller ordinal $P_n(\alpha)$. Hence part 1 of the theorem, by well-foundedness.

E.G. With $\alpha = \omega^{\omega^\omega}$ and $n = 2$ we get $a = 2^{2^2} = 16$.
Then $a - 1 = 2^{2+1} + 2^2 + 2^1 + 1$ and $P_n(\alpha) = \omega^{\omega+1} + \omega^\omega + \omega^1 + 1$.

Definition

$$H_0(n) \;=\; n \quad \text{and} \quad H_\alpha(n) \;=\; H_{P_n(\alpha)}(n+1)\,.$$

Theorem (Cichon (1983))

$H_\alpha(n)$ measures the length of any Goodstein sequence on a, n.
A proof that all G-sequences terminate says H_{ε_0} is recursive.
But H_{ε_0} is not provably recursive in PA. Hence part 2.

§1.2. The Fast Growing Hierarchy $B_\alpha := H_{2^\alpha}$

- $H_{\alpha+\beta} = H_\alpha \circ H_\beta$.
- So $B_0 =$ successor and $B_{\alpha+1} = B_\alpha \circ B_\alpha$.
  Thus, for finite α, $B_\alpha(n) = n + 2^\alpha$.
- This is the beginning of the (sometimes called "Schwichtenberg - Wainer") Fast Growing Hierarchy:

$$B_\alpha(n) = \begin{cases} n+1 & \text{if } \alpha = 0 \\ B_{\alpha'}(B_{\alpha'}(n)) & \text{if } \alpha = \alpha' + 1 \\ B_{\alpha_n}(n) & \text{if } \alpha \text{ is a limit} \end{cases}$$

where $\{\alpha_n\}_{n \in N}$ is a chosen fundamental sequence to limit α.

Theorem

$\{B_\alpha\}_{\alpha<|T|}$ classifies provable recursion in arithmetical theories T, i.e. provides bounds for witnesses of provable Σ_1 formulas.

§2. Input-Output Arithmetic

Definition

- EA(I;O) has the language of arithmetic, with (quantified, "output") variables $a, b, \ldots$.
- It has symbols for arbitrary (partial) recursive functions, given by their defining equations.
- In addition there are numerical constants ("inputs") $x, y, \ldots$.
- *Basic terms* are those built from the constants and variables by successive application of successor and predecessor.
- Only basic terms are allowed as "witnesses" in the logical rules for $\forall$ and $\exists$.
- The induction axioms are:

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \;\rightarrow\; A(t(x))$$

where t is a closed, basic term controlling induction-length!

§2.1. Working in EA(I;O)

Definition

Write $t \downarrow$ for $\exists a(t = a)$.
Note that if t is not basic one cannot pass directly from $t = t$ to $t \downarrow$. But $a + 1$ is basic, so $t \downarrow \rightarrow t + 1 \downarrow$.

In general, $t \downarrow \wedge A(t) \rightarrow \exists a A(a)$ and dually for $\forall$.

Example

- From $b + c \downarrow$ we then immediately get $b + (c + 1) \downarrow$.
  Therefore $b + x \downarrow$ by Σ_1-induction "up to" x.
  Then $\forall b(b + x \downarrow)$.
- By iterating this x times, we thus obtain $b + x^2 \downarrow$.
  Then $\forall b(b + x^2 \downarrow)$, $\forall b(b + x^3 \downarrow)$. Etc.
- Exponential requires a Π_2 induction on $\forall b(b + 2^c \downarrow)$.

Proving $\forall b(b + 2^x \downarrow)$ with Π_2 induction - an argument going back to Gentzen

Assume $\forall b(b + 2^c \downarrow)$.

Then, given an arbitrary b, we have:

$b + 2^c \downarrow$ by the assumption, and hence

$(b + 2^c) + 2^c \downarrow$ by the assumption again.

Therefore $\forall b(b + 2^c \downarrow) \rightarrow \forall b(b + 2^{c+1} \downarrow)$,
and $\forall b(b + 2^0 \downarrow)$ because $b + 1$ is basic.

Therefore $\mathrm{I}\Pi_2(\mathrm{I};\mathrm{O}) \vdash \forall b(b + 2^x \downarrow)$.

Similarly $\mathrm{I}\Pi_2(\mathrm{I};\mathrm{O}) \vdash \forall b(b + 2^{p(\vec{x})} \downarrow)$.

Then $\mathrm{I}\Pi_3(\mathrm{I};\mathrm{O}) \vdash \forall b(b + 2^{2^x} \downarrow)$ etcetera.

§2.2. Bounding Σ_1-Inductions

Theorem

Witnesses for Σ_1 theorems, proved by Σ_1-inductions up to $x := n$, are bounded by B_h where $h = \log n$.

Proof.

- Any induction up to $x := n$ can be unravelled, inside EA(I;O), to a binary tree of Cuts (modus ponens) of height $\log n$.
- If it's a Σ_1-induction on $\exists a A(b, a)$ and we assume that B_h bounds witnesses at height h, then a further Cut:

$$\frac{\exists a A(b, a) \to \exists a A(b', a) \quad \exists a A(b', a) \to \exists a A(b'', a)}{\exists a A(b, a) \to \exists a A(b'', a)}$$

will yield a bound $B_{h+1} = B_h \circ B_h$ at height $h + 1$.

$\square$

§2.3. Provable Functions of EA(I;O)

Definition

A provably recursive function of EA(I;O) is one which is provably defined on inputs, i.e. $\vdash f(x)\downarrow$.

Theorem (Ostrin-W. APAL 133, 2005)

The provable functions of $I\,\Sigma_1(I;O)$ are those computable in poly-time $p(n)$.
The provable functions of $I\,\Sigma_2(I;O)$ are those computable in exp-time $2^{p(n)}$. Etcetera, up the Ritchie-Schwichtenberg hierarchy of all elementary functions.

(Cf. Leivant's Ramified Inductions (1995) where such characterizations were first obtained. Also Nelson's Predicative Arithmetic (1986).)

§2.4. Proof

Proof.

- Fix $x := n$ in $\vdash f(x) \downarrow$ say with d nested inductions.
- Partial cut-elimination yields a "free-cut-free" proof, so only "inductive cuts" remain.
- The height of the proof-tree will be (of the order of) $\log n \cdot d$.
- For $\mathsf{I}\Sigma_1(\mathsf{I};\mathsf{O})$ the Bounding Theorem applies immediately to give complexity bounds

$$B_{\log n \cdot d}(0) = 0 + 2^{\log n \cdot d} = n^d.$$

- For $\mathsf{I}\Sigma_2$ one must first reduce all cuts to Σ_1 form by Gentzen cut-reduction, which further increases the height by an exponential, so in that case the complexity bounds will be

$$B_{2^{\log n \cdot d}}(0) = 2^{n^d}.$$

$\square$

§3. Adding an Inductive Definition

Definition

ID_1(I;O) is obtained from EA(I;O) by adding, for each positive inductive form $F(X, a)$, a new predicate P, and Closure and Least-Fixed-Point axioms:

$$\forall a(F(P, a) \to P(a))$$

$$\forall a(F(A, a) \to A(a)) \to \forall a(P(a) \to A(a))$$

for each formula A.

Example

- Associate the predicate N with the inductive form:

$$F(X, a) \;:\equiv\; a = 0 \vee \exists b(X(b) \wedge a = b + 1).$$

§3.1. Embedding Peano Arithmetic

Theorem

If $PA \vdash A$ then $ID_1(I;O) \vdash A^N$.

Proof.

- From the axioms one easily deduces:

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \;\rightarrow\; \forall a(F(A,a) \rightarrow A(a))$$

- Therefore:

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \;\rightarrow\; \forall a(N(a) \rightarrow A(a)).$$

- Hence Peano Arithmetic is interpreted in $\mathrm{ID}_1(\mathrm{I};\mathrm{O})$ by relativizing quantifiers to N.

$\square$

§3.2. Unravelling LFP-Ax by Buchholz' Ω-Rule

We are still working in the I/O context, so can fix $\vec{x} := \vec{n}$ and unravel inductions into iterated Cuts as before.

However the resulting $\mathrm{ID}_1(\mathrm{I};\mathrm{O})$-derivations will be further complicated by the presence of Least-Fixed-Point axioms.

These must be "unravelled" as well, before we can read off bounds.

Definition (Buchholz' Ω-Rule)

$$\frac{\vdash^{\lambda_0} N(m), \Gamma_0 \qquad \vdash_0^h N(m) \Rightarrow \vdash^{\lambda_h} \Gamma_1}{\vdash^{\lambda} \Gamma_0, \Gamma_1}$$

The map $h \mapsto \lambda_h$ measures the transformation's uniformity. Hence ordinal heights now appear as the infinitary content.

§3.3. Ω Proves LFP-Ax

The gist of it.

- For the left-hand premise of the Ω-rule choose $\vdash^0 N(m), \neg N(m)$.
- For the right-hand premise, first assume $\vdash_0^h N(m)$.
- Each step of this (direct, cut-free) proof can be mimicked to prove $A(m)$ if we assume that A is "progressive".
- Thus $\vdash^h \neg\forall a(F(A,a) \to A(a)), A(m)$.
- The standard fundamental sequence for ω gives $\omega_h = h$.
- Ω-rule gives $\vdash^\omega \neg\forall a(F(A,a) \to A(a)), \neg N(m), A(m)$ for every m.
- Then by ω-rule obtain LFP-Ax with proof-height $\omega + 3$.

$\square$

§3.4. Cut Elimination and Collapsing in $\mathrm{ID}_1(\mathrm{I};\mathrm{O})^\infty$

As usual, Gentzen-style cut-reduction raises height exponentially.

Lemma (Collapsing)

If, for fixed input $x := n$, we have a cut-free derivation $\vdash_0^\alpha \Gamma$ with Γ positive in N, then $\vdash_0^k \Gamma$ where $k \leq B_{\alpha+1}(n)$.

Proof.

- For Ω-rule, assume it holds for the premises

$$\vdash_0^{\alpha_0} N(m), \Gamma_0 \text{ and } \vdash_0^h N(m), \Gamma_0 \ \Rightarrow \vdash_0^{\alpha_h} \Gamma.$$

- Then for the left premise, $\vdash_0^h N(m), \Gamma_0$ where $h \leq B_{\alpha_0+1}(n)$.
- And for the right premise, $\vdash_0^k \Gamma$ where $k \leq B_{\alpha_h+1}(n)$.
- Hence $k \leq B_{\alpha_h+1}(n) \leq B_\alpha(h+1) \leq B_\alpha B_\alpha(n) = B_{\alpha+1}(n)$.

$\square$

§3.5. A "New" Proof of an Old Theorem

Theorem

Every Σ_1 theorem of PA has witnesses bounded by B_α for some $\alpha < \varepsilon_0$.

Proof.

- Embed as $\mathsf{ID}_1(\mathsf{I};\mathsf{O}) \vdash \exists a(N(a) \wedge A(x,a))$ with x input.
- Translate this into $\mathsf{ID}_1(\mathsf{I};\mathsf{O})^\infty$ with proof-height $\omega + k$, cut-rank r.
- Eliminate cuts to obtain proof-height $\alpha = 2_r(\omega + k) < \varepsilon_0$.
- Collapse to obtain $\vdash^h \exists a(N(a) \wedge A(x,a))$ with $h = B_{\alpha+1}(x)$.
- Use original Bounding Theorem to bound witness a below $B_h(x) \leq B_\omega B_{\alpha+1}(x) \leq B_{\alpha+2}(x)$.

$\square$

§4. Generalizing to $\mathrm{ID}_{<\omega}$

Williams' thesis (Leeds 2004) generalizes the foregoing to theories of finitely iterated inductive definitions $\mathrm{ID}_i(\mathrm{I};\mathrm{O})$. As the inductive definitions are iterated, the higher levels of Ω-rules needed to unravel them require ordinals in successively higher number-classes $\Omega_1, \Omega_2, \ldots, \Omega_i$.

Collapsing (and Bounding) from one level $i+1$ down to the one below is then computed in terms of higher-level extensions of the B_α hierarchy: $\varphi^{(i)}_\alpha(\beta)$ for $\alpha \in \Omega_{i+1}, \beta \in \Omega_i$ where $\varphi(\beta)$ starts with $\beta+1$, composes at successors, diagonalizes at "big" limits, (just like B) and is continuously extended at small limits.

The ordinal bound of $\mathrm{ID}_2(\mathrm{I};\mathrm{O})$ is then the Bachmann-Howard:

$$\tau_3 \;=\; \varphi^{(1)}_{\varphi^{(2)}_{\varphi^{(3)}_\omega(\omega_2)}(\omega_1)}(\omega).$$

§4.1. Bounding Functions for $\mathrm{ID}_{<\omega}$ and Π^1_1-CA_0

Define $\varphi^{(k)} : \Omega_{k+1} \times \Omega_k \to \Omega_k$ by:

$$\varphi^{(k)}_\alpha(\beta) = \begin{cases} \beta + 1 & \text{if } \alpha = 0 \\ \varphi^{(k)}_\gamma \circ \varphi^{(k)}_\gamma(\beta) & \text{if } \alpha = \gamma + 1 \\ \varphi^{(k)}_{\alpha_\beta}(\beta) & \text{if } \alpha = \sup \alpha_\xi \ (\xi \in \Omega_k) \\ \sup \varphi^{(k)}_{\alpha_\xi}(\beta) & \text{if } \alpha = \sup \alpha_\xi \ (\xi \in \Omega_{<k}) \end{cases}$$

Define $\tau = \sup \tau_i$ where $\tau_0 = \omega$ and

$$\tau_1 = \varphi^{(1)}_\omega(\omega)\ ,\ \tau_2 = \varphi^{(1)}_{\varphi^{(2)}_\omega(\omega_1)}(\omega)\ ,\ \tau_3 = \varphi^{(1)}_{\varphi^{(2)}_{\varphi^{(3)}_\omega(\omega_2)}(\omega_1)}(\omega)\ ,\ \ldots$$

Theorem

The proof-theoretic ordinal of ID_i is τ_{i+2}. The provably computable functions of Π^1_1-CA_0 are those computably-bounded by $\{B_\alpha\}_{\alpha \prec \tau}$.

§5. Links to Independence Results

Theorem (Friedman's Miniaturized Kruskal Theorem for Labelled Trees)

For each constant c there is a number $K(c)$ so large that in every sequence $\{T_j\}_{j<K(c)}$ of finite trees with labels from a given finite set, and such that $|T_j| \leq c \cdot 2^j$, there are $j_1 < j_2$ such that $T_{j_1} \hookrightarrow T_{j_2}$. The embedding must preserve infs, labels, and satisfy a certain "gap condition".

Lemma

The (natural) computation sequence for $B_{\tau_i}(n)$ satisfies the size-bound above, and is a bad sequence, i.e. no embeddings.

Corollary

For a simple c_n we therefore have $B_{\tau_n}(n) < K(c_n)$ for all n. Therefore K is not provable recursive in $ID_{<\omega}$, nor in Π^1_1-CA_0.

§5.1. The Computation Sequence for τ_n

By reducing/rewriting τ_n according to the defining equations of the φ-functions, we pass through all the ordinals in $\tau_n\,[n]$. Each term is a binary tree with labels $\leq n$, and each one-step-reduction at most doubles the size of the tree. E.g. with $n = 2$ the sequence begins:

$$\tau_2 \to \varphi^{(1)}_{\varphi^{(2)}_2(\omega_1)}(\omega) \to \varphi^{(1)}_{\varphi^{(2)}_1\varphi^{(2)}_1(\omega_1)}(\omega) \to \varphi^{(1)}_{\varphi^{(2)}_0\varphi^{(2)}_0\varphi^{(2)}_1(\omega_1)}(\omega) \to$$

$$\varphi^{(1)}_{\varphi^{(2)}_0\varphi^{(2)}_1(\omega_1)}(\varphi^{(1)}_{\varphi^{(2)}_0\varphi^{(2)}_1(\omega_1)}(\omega)) \to \varphi^{(1)}_{\varphi^{(2)}_1(\omega_1)}(\varphi^{(1)}_{\varphi^{(2)}_1(\omega_1)}(\varphi^{(1)}_{\varphi^{(2)}_0\varphi^{(2)}_1(\omega_1)}(\omega)))$$

$$\to \varphi^{(1)}_{\omega_1}\varphi^{(1)}_{\omega_1}\varphi^{(1)}_{\varphi^{(2)}_0(\omega_1)}\varphi^{(1)}_{\varphi^{(2)}_1(\omega_1)}\varphi^{(1)}_{\varphi^{(2)}_0\varphi^{(2)}_1(\omega_1)}(\omega) \to \varphi^{(1)}_{\varphi^{(1)}_{\omega_1}(-)}(\varphi^{(1)}_{\omega_1}(-))\ldots$$

The length of the entire sequence (down to zero) is therefore $\geq |\tau_n\,[n]| = G_n(\tau_n) = B_{\tau_{n-1}}(n)$. Furthermore, the sequence is bad - no term is gap-embeddable in any follower.