Fast Growing Functions and Arithmetical Independence Results

Stanley S. Wainer (Leeds UK)

Stanford, March 2013
Are there any genuine mathematical examples of incompleteness?


Take any number \(a\), for example \(a = 16\).

Write \(a\) in “complete base-2”, thus \(a = 2^2^2\).

Subtract 1, so the base-2 representation is \(a - 1 = 2^{2+1} + 2^2 + 2^1 + 1\).

Increase the base by 1, to produce the next stage \(a_1 = 3^{3+1} + 3^3 + 3^1 + 1 = 112\).

Continue subtracting 1 and increasing the base: \(a, a_1, a_2, a_3, \ldots\). Example: 16, 112, 1284, 18753, 326594, ...

Theorem (Part 1 – Goodstein, Part 2 – Kirby & Paris)

(1) Every Goodstein sequence eventually terminates in 0.
(2) But this is not provable in Peano Arithmetic (PA).
§1.1. The “Hardy” Functions

If, throughout any Cantor Normal Form $\alpha \prec \varepsilon_0$, we replace $\omega$ by a large enough $n$, then we obtain a “complete base-n” representation. Subtracting 1, and then putting the $\omega$ back, one gets a smaller ordinal $P_n(\alpha)$. Hence part 1 of the theorem, by well-foundedness.

E.G. With $\alpha = \omega^{\omega^{\omega}}$ and $n = 2$ we get $a = 2^{2^2} = 16$. Then $a - 1 = 2^{2+1} + 2^2 + 2^1 + 1$ and $P_n(\alpha) = \omega^{\omega+1} + \omega^{\omega} + \omega^1 + 1$.

Definition

$$H_0(n) = n \text{ and } H_{\alpha}(n) = H_{P_n(\alpha)}(n + 1).$$

Theorem (Cichon (1983))

$H_{\alpha}(n)$ measures the length of any Goodstein sequence on $a, n$. A proof that all G-sequences terminate says $H_{\varepsilon_0}$ is recursive. But $H_{\varepsilon_0}$ is not provably recursive in PA. Hence part 2.
§1.2. The Fast Growing Hierarchy $B_\alpha := H_{2^\alpha}$

- $H_{\alpha+\beta} = H_\alpha \circ H_\beta$.
- So $B_0 = \text{successor}$ and $B_{\alpha+1} = B_\alpha \circ B_\alpha$.
  Thus, for finite $\alpha$, $B_\alpha(n) = n + 2^\alpha$.
- This is the beginning of the (sometimes called “Schwichtenberg - Wainer”) Fast Growing Hierarchy:

$$B_\alpha(n) = \begin{cases} 
  n + 1 & \text{if } \alpha = 0 \\
  B_{\alpha'}(B_{\alpha'}(n)) & \text{if } \alpha = \alpha' + 1 \\
  B_{\alpha_n}(n) & \text{if } \alpha \text{ is a limit}
\end{cases}$$

where $\{\alpha_n\}_{n \in \mathbb{N}}$ is a chosen fundamental sequence to limit $\alpha$.

**Theorem**

$\{B_\alpha\}_{\alpha < |T|}$ classifies provable recursion in arithmetical theories $T$,
i.e. provides bounds for witnesses of provable $\Sigma_1$ formulas.
§2. Input-Output Arithmetic

Definition

- EA(I;O) has the language of arithmetic, with (quantified, “output”) variables $a, b, \ldots$.
- It has symbols for arbitrary (partial) recursive functions, given by their defining equations.
- In addition there are numerical constants (“inputs”) $x, y, \ldots$.
- *Basic terms* are those built from the constants and variables by successive application of successor and predecessor.
- Only basic terms are allowed as “witnesses” in the logical rules for $\forall$ and $\exists$.
- The induction axioms are:

\[
A(0) \land \forall a (A(a) \rightarrow A(a + 1)) \rightarrow A(t(x))
\]

where $t$ is a closed, basic term controlling induction-length!
2.1. Working in EA(I;O)

Definition

Write \( t \downarrow \) for \( \exists a (t = a) \).

Note that if \( t \) is not basic one cannot pass directly from \( t = t \) to \( t \downarrow \). But \( a + 1 \) is basic, so \( t \downarrow \rightarrow t + 1 \downarrow \).

In general, \( t \downarrow \wedge A(t) \rightarrow \exists a A(a) \) and dually for \( \forall \).

Example

- From \( b + c \downarrow \) we then immediately get \( b + (c + 1) \downarrow \). Therefore \( b + x \downarrow \) by \( \Sigma_1 \)-induction “up to” \( x \).
  Then \( \forall b (b + x \downarrow) \).

- By iterating this \( x \) times, we thus obtain \( b + x^2 \downarrow \).
  Then \( \forall b (b + x^2 \downarrow), \forall b (b + x^3 \downarrow) \). Etc.

- Exponential requires a \( \Pi_2 \) induction on \( \forall b (b + 2^c \downarrow) \).
Proving $\forall b (b + 2^x \downarrow)$ with $\Pi_2$ induction - an argument going back to Gentzen

Assume $\forall b (b + 2^c \downarrow)$.

Then, given an arbitrary $b$, we have:

$b + 2^c \downarrow$ by the assumption, and hence

$(b + 2^c) + 2^c \downarrow$ by the assumption again.

Therefore $\forall b (b + 2^c \downarrow) \rightarrow \forall b (b + 2^{c+1} \downarrow)$,

and $\forall b (b + 2^0 \downarrow)$ because $b + 1$ is basic.

Therefore $\Pi_2(I;O) \vdash \forall b (b + 2^x \downarrow)$.

Similarly $\Pi_2(I;O) \vdash \forall b (b + 2^{p(\vec{x})} \downarrow)$.

Then $\Pi_3(I;O) \vdash \forall b (b + 2^{2^x} \downarrow)$ etcetera.
§2.2. Bounding $\Sigma_1$-Inductions

Theorem
Witnesses for $\Sigma_1$ theorems, proved by $\Sigma_1$-inductions up to $x := n$, are bounded by $B_h$ where $h = \log n$.

Proof.

- Any induction up to $x := n$ can be unravelled, inside EA(I;O), to a binary tree of Cuts (modus ponens) of height $\log n$.
- If it’s a $\Sigma_1$-induction on $\exists aA(b, a)$ and we assume that $B_h$ bounds witnesses at height $h$, then a further Cut:

$$\exists aA(b, a) \rightarrow \exists aA(b', a) \quad \exists aA(b', a) \rightarrow \exists aA(b'', a)$$

$$\exists aA(b, a) \rightarrow \exists aA(b'', a)$$

will yield a bound $B_{h+1} = B_h \circ B_h$ at height $h + 1$. 

□
2.3. Provable Functions of EA(I;O)

Definition
A provably recursive function of EA(I;O) is one which is provably defined on inputs, i.e. $\vdash f(x) \downarrow$.

Theorem (Ostrin-W. APAL 133, 2005)
The provable functions of $I \Sigma_1(I;O)$ are those computable in poly-time $p(n)$.
The provable functions of $I \Sigma_2(I;O)$ are those computable in exp-time $2^{p(n)}$. Etcetera, up the Ritchie-Schwichtenberg hierarchy of all elementary functions.

(Cf. Leivant’s Ramified Inductions (1995) where such characterizations were first obtained. Also Nelson’s Predicative Arithmetic (1986).)
§2.4. Proof

Proof.

- Fix $x := n$ in $\vdash f(x) \downarrow$ say with $d$ nested inductions.
- Partial cut-elimination yields a “free-cut-free” proof, so only “inductive cuts” remain.
- The height of the proof-tree will be (of the order of) $\log n \cdot d$.
- For $I\Sigma_1(I;O)$ the Bounding Theorem applies immediately to give complexity bounds

$$B_{\log n \cdot d}(0) = 0 + 2^{\log n \cdot d} = n^d.$$  

- For $I\Sigma_2$ one must first reduce all cuts to $\Sigma_1$ form by Gentzen cut-reduction, which further increases the height by an exponential, so in that case the complexity bounds will be

$$B_{2\log n \cdot d}(0) = 2^{n^d}.$$
3. Adding an Inductive Definition

Definition

ID₁(I;O) is obtained from EA(I;O) by adding, for each positive inductive form \( F(X, a) \), a new predicate \( P \), and Closure and Least-Fixed-Point axioms:

\[
\forall a (F(P, a) \rightarrow P(a))
\]

\[
\forall a (F(A, a) \rightarrow A(a)) \rightarrow \forall a (P(a) \rightarrow A(a))
\]

for each formula \( A \).

Example

- Associate the predicate \( N \) with the inductive form:

\[
F(X, a) \equiv a = 0 \lor \exists b (X(b) \land a = b + 1).
\]
§3.1. Embedding Peano Arithmetic

Theorem

If $PA \vdash A$ then $ID_1(I;O) \vdash A^N$.

Proof.

- From the axioms one easily deduces:

$$A(0) \land \forall a(A(a) \rightarrow A(a + 1)) \rightarrow \forall a(F(A, a) \rightarrow A(a))$$

- Therefore:

$$A(0) \land \forall a(A(a) \rightarrow A(a + 1)) \rightarrow \forall a(N(a) \rightarrow A(a)).$$

- Hence Peano Arithmetic is interpreted in $ID_1(I;O)$ by relativizing quantifiers to $N$. □
§3.2. Unravelling LFP-Ax by Buchholz’ Ω-Rule

We are still working in the I/O context, so can fix \( \vec{x} := \vec{n} \) and unravel inductions into iterated Cuts as before.

However the resulting ID\(_1\)(I;O)-derivations will be further complicated by the presence of Least-Fixed-Point axioms. These must be “unravelled” as well, before we can read off bounds.

**Definition (Buchholz’ Ω-Rule)**

\[
\vdash^\lambda_0 N(m), \Gamma_0 \quad \vdash^h_0 N(m) \Rightarrow \vdash^\lambda \Gamma_0, \Gamma_1
\]

The map \( h \mapsto \lambda_h \) measures the transformation’s uniformity. Hence ordinal heights now appear as the infinitary content.
The gist of it.

- For the left-hand premise of the $\Omega$-rule choose $\vdash^0 N(m), \neg N(m)$.
- For the right-hand premise, first assume $\vdash_h^0 N(m)$.
- Each step of this (direct, cut-free) proof can be mimicked to prove $A(m)$ if we assume that $A$ is “progressive”.
- Thus $\vdash^h \neg \forall a (F(A, a) \rightarrow A(a)), A(m)$.
- The standard fundamental sequence for $\omega$ gives $\omega_h = h$.
- $\Omega$-rule gives $\vdash^\omega \neg \forall a (F(A, a) \rightarrow A(a)), \neg N(m), A(m)$ for every $m$.
- Then by $\omega$-rule obtain LFP-Ax with proof-height $\omega + 3$. 
As usual, Gentzen-style cut-reduction raises height exponentially.

**Lemma (Collapsing)**

If, for fixed input \( x := n \), we have a cut-free derivation \( \Gamma \) with \( \Gamma \) positive in \( N \), then \( \Gamma \) where \( k \leq B_{\alpha+1}(n) \).

**Proof.**

- For \( \Omega \)-rule, assume it holds for the premises

\[
\vdash_{\alpha_0}^\alpha N(m), \Gamma_0 \text{ and } \vdash_{h_0}^h N(m), \Gamma_0 \Rightarrow \vdash_{\alpha_0}^{\alpha_h} \Gamma.
\]

- Then for the left premise, \( \vdash_{h_0}^h N(m), \Gamma_0 \) where \( h \leq B_{\alpha_0+1}(n) \).
- And for the right premise, \( \vdash_{k_0}^k \Gamma \) where \( k \leq B_{\alpha_{h+1}}(n) \).
- Hence \( k \leq B_{\alpha_{h+1}}(n) \leq B_{\alpha}(h + 1) \leq B_{\alpha}B_{\alpha}(n) = B_{\alpha+1}(n). \)

\( \blacksquare \)
§3.5. A “New” Proof of an Old Theorem

**Theorem**

*Every $\Sigma_1$ theorem of $PA$ has witnesses bounded by $B_\alpha$ for some $\alpha < \varepsilon_0$.***

**Proof.**

- Embed as $\text{ID}_1(I;O) \vdash \exists a(N(a) \land A(x,a))$ with $x$ input.
- Translate this into $\text{ID}_1(I;O)^\infty$ with proof-height $\omega + k$, cut-rank $r$.
- Eliminate cuts to obtain proof-height $\alpha = 2r(\omega + k) < \varepsilon_0$.
- Collapse to obtain $\vdash^h \exists a(N(a) \land A(x,a))$ with $h = B_{\alpha+1}(x)$.
- Use original Bounding Theorem to bound witness $a$ below $B_h(x) \leq B_\omega B_{\alpha+1}(x) \leq B_{\alpha+2}(x)$. 

$\square$
Williams’ thesis (Leeds 2004) generalizes the foregoing to theories of finitely iterated inductive definitions $\text{ID}_i(I;O)$. As the inductive definitions are iterated, the higher levels of $\Omega$-rules needed to unravel them require ordinals in successively higher number-classes $\Omega_1, \Omega_2, \ldots, \Omega_i$.

Collapsing (and Bounding) from one level $i + 1$ down to the one below is then computed in terms of higher-level extensions of the $B_\alpha$ hierarchy: $\varphi_{\alpha}^{(i)}(\beta)$ for $\alpha \in \Omega_{i+1}, \beta \in \Omega_i$ where $\varphi(\beta)$ starts with $\beta + 1$, composes at successors, diagonalizes at “big” limits, (just like $B$) and is continuously extended at small limits.

The ordinal bound of $\text{ID}_2(I;O)$ is then the Bachmann-Howard:

$$\tau_3 = \varphi_{\varphi_\omega(3)}^{(1)}(\omega_2)(\omega_1)(\omega).$$
§4.1. Bounding Functions for ID$_{<\omega}$ and $\Pi^1_1$-CA$_0$

Define $\varphi^{(k)} : \Omega_{k+1} \times \Omega_k \rightarrow \Omega_k$ by:

$$
\varphi^{(k)}_{\alpha} (\beta) = \begin{cases} 
\beta + 1 & \text{if } \alpha = 0 \\
\varphi^{(k)}_{\gamma} \circ \varphi^{(k)}_{\gamma} (\beta) & \text{if } \alpha = \gamma + 1 \\
\varphi^{(k)}_{\alpha \beta} (\beta) & \text{if } \alpha = \sup \alpha_\xi (\xi \in \Omega_k) \\
\sup \varphi^{(k)}_{\alpha \xi} (\beta) & \text{if } \alpha = \sup \alpha_\xi (\xi \in \Omega_{<k})
\end{cases}
$$

Define $\tau = \sup \tau_i$ where $\tau_0 = \omega$ and

$$
\tau_1 = \varphi^{(1)}_{\omega} (\omega) , \tau_2 = \varphi^{(1)}_{\varphi^{(2)}_{\omega}(\omega_1)} (\omega) , \tau_3 = \varphi^{(1)}_{\varphi^{(3)}_{\omega}(\omega_2)} (\omega_1) (\omega) , \ldots
$$

Theorem

The proof-theoretic ordinal of ID$_i$ is $\tau_{i+2}$. The provably computable functions of $\Pi^1_1$-CA$_0$ are those computably-bounded by $\{ B_\alpha \}_{\alpha < \tau}$. 
§5. Links to Independence Results

Theorem (Friedman’s Miniaturized Kruskal Theorem for Labelled Trees)

For each constant $c$ there is a number $K(c)$ so large that in every sequence $\{ T_j \}_{j < K(c)}$ of finite trees with labels from a given finite set, and such that $|T_j| \leq c \cdot 2^j$, there are $j_1 < j_2$ such that $T_{j_1} \hookrightarrow T_{j_2}$. The embedding must preserve infs, labels, and satisfy a certain “gap condition”.

Lemma

The (natural) computation sequence for $B_{\tau_i}(n)$ satisfies the size-bound above, and is a bad sequence, i.e. no embeddings.

Corollary

For a simple $c_n$ we therefore have $B_{\tau_n}(n) < K(c_n)$ for all $n$. Therefore $K$ is not provable recursive in $ID_{<\omega}$, nor in $\Pi^1_1$-$CA_0$. 
§5.1. The Computation Sequence for $\tau_n$

By reducing/rewriting $\tau_n$ according to the defining equations of the $\varphi$-functions, we pass through all the ordinals in $\tau_n [n]$. Each term is a binary tree with labels $\leq n$, and each one-step-reduction at most doubles the size of the tree. E.g. with $n = 2$ the sequence begins:

$\tau_2 \rightarrow \varphi_2^{(2)}(\omega_1)(\omega) \rightarrow \varphi_1^{(1)} \varphi_1^{(2)}(\omega_1)(\omega) \rightarrow \varphi_0^{(1)} \varphi_1^{(2)}(\omega_1)(\omega)$

$\varphi_0^{(2)} \varphi_1^{(2)}(\omega_1)(\varphi_0^{(1)} \varphi_1^{(2)}(\omega_1)(\omega))) \rightarrow \varphi_0^{(2)} \varphi_1^{(2)}(\omega_1)(\varphi_0^{(1)} \varphi_1^{(2)}(\omega_1)(\omega)))$

$\varphi_0^{(2)} \varphi_1^{(2)}(\omega_1)(\varphi_0^{(1)} \varphi_1^{(2)}(\omega_1)(\omega)) \rightarrow \varphi_0^{(1)} \varphi_1^{(2)}(\omega_1)(\omega) \rightarrow \varphi_0^{(1)} \varphi_1^{(2)}(\omega_1)(\omega)$

The length of the entire sequence (down to zero) is therefore

$\geq |\tau_n [n]| = G_n(\tau_n) = B_{\tau_{n-1}}(n)$. Furthermore, the sequence is bad - no term is gap-embeddable in any follower.