The λ-calculus can be put to work as Mathematics, not Formalism!

Dana S, Scott

30 October 2012

And the ideas of many people can be usefully and simply recycled!

(1) The powerset space **P** with its weak topology is ***universal*** for all countably based T_0-spaces. That is, the subsets of **P** inherit from **P** a subspace topology — and ***all*** the relevant T_0-spaces turn up this way.

(2) The ***injective properties*** of **P** guarantee that any continuous function F: X $\longrightarrow$ Y between subspaces is the restriction of a continuous function G: **P** $\longrightarrow$ **P**. That is, ***continuity*** of functions is inherited as well as topology.

(3) The λ-calculus on P gives a ***notation*** for defining continuous functions with broad, general properties. There is also a canonical representation of the function space (**P** $\longrightarrow$ **P**) as a subspace of **P**.

(4) The ***Fixed-Point Theorem*** for functions G: **P** $\longrightarrow$ **P** and its definition in λ-calculus, together with the arithmetic combinators, give us a way of defining all the computable enumeration operators and all r.e. subsets of **P**. ***Enumeration degrees*** thus become a "chapter" of λ-calculus.

(5) The subspaces X of **P** and the continuous mappings F: X $\longrightarrow$ X also inherit notions of ***computability*** from **P**. In this way we can speak of ***degrees*** contained in X. In particular, for the subspace (N $\longrightarrow$ N) of **P** we get the usual ***Turing degrees.***

(6) In order to pass to ***higher types***, it is best to consider ***quotients*** of subspaces of P. The simplest way to do this is to use ***partial equivalence relations*** (PERs) on P. The PERs form a ***cartesian closed category***, and all the PERs inherit both topology and computability from P. ***WARNING:*** The PERs are not in themselves topologically defined, being arbitrary quotients, nor does topology determine the function spaces in general. ***EXAMPLE:*** Let N be the identity relation on the (singletons of the) integers. Then the types as PERs: ((. . .((N $\longrightarrow$ N) $\longrightarrow$ N) . . . $\longrightarrow$ N) $\longrightarrow$ N) give us the well known ***Kleene-Kreisel Countable Functionals***.

(7) Operations on PERs can be expanded to include ***products*** and ***sums*** of dependent type systems. Invoking the ***propositions- as-types*** paradigm gives us a modeling of Martin-Löf Type Theory which additionally has computability as a ***modal operator*** (#).

(8) By combining Type Theory and Computability Theory, we have a way of doing (much more than) ***Bishop-style Constructive Analysis*** in a definite model universe — rather than as a formal (intuitionistic) theory. ***WARNING:*** The topological reals R_t as a subspace of P is OK, but it is better to use the Cauchy reals R_c as Bishop does. We can map R_c onto R_t, but there is no inverse mapping, since there is no way to pick a Cauchy sequence for each real in a continuous way. Dag Normann has found it a hard study to understand the higher types $((\ldots((R_c \longrightarrow R_c) \longrightarrow R_c) \ldots \longrightarrow R_c) \longrightarrow R_c)$.

(9) The simple and natural ***embedding*** of familiar structures as types over **P** ought to allow us to incorporate much past work about computable structures (e.g., algebras) into work in this model. And we can work easily with ***higher types***.

(10) ***Recursive isomorphism*** of types ought to lead to new problems in a new "cardinal arithmetic". Older work of Myhill, Nerode & Co. should be directly relevant.

(11) Other notions of computability can be introduced by going further up the ***Kleene Hierarchy*** — and even into to the ***Constructible Hierarchy*** as done in Admissible Set Theory. The underlying universe **P** has only the cardinality of the continuum, but it is a very rich universe.

(12) By using ***intersections of types*** and ***type polymorphism***, it is possible that many ideas of general computability could be ***defined*** with a quite limited suite of logical notions.

(13) Even though the facts were presented ***semantically*** *via* a special kind of modeling on **P**, once some general principles are established, further ***proofs*** can be done formally in the style of Bishop Constructive Analysis or of Martin-Löf Type Theory.

A FINAL WARNING: There is all too often a long gap between an initial ***definition*** and a complete ***analysis*** of a concept. But simple, well structured definitions in a uniform framework ought to make the bridging of gaps easier and more understandable.