The λ-calculus can be put to work as Mathematics, not Formalism!

Dana S, Scott
30 October 2012

And the ideas of many people can be usefully and simply recycled!

(1) The powerset space $P$ with its weak topology is universal for all countably based $T_0$-spaces. That is, the subsets of $P$ inherit from $P$ a subspace topology — and all the relevant $T_0$-spaces turn up this way.

(2) The injective properties of $P$ guarantee that any continuous function $F: X \rightarrow Y$ between subspaces is the restriction of a continuous function $G: P \rightarrow P$. That is, continuity of functions is inherited as well as topology.

(3) The λ-calculus on $P$ gives a notation for defining continuous functions with broad, general properties. There is also a canonical representation of the function space $(P \rightarrow P)$ as a subspace of $P$.

(4) The Fixed-Point Theorem for functions $G: P \rightarrow P$ and its definition in λ-calculus, together with the arithmetic combinators, give us a way of defining all the computable enumeration operators and all r.e. subsets of $P$. Enumeration degrees thus become a "chapter" of λ-calculus.

(5) The subspaces $X$ of $P$ and the continuous mappings $F: X \rightarrow X$ also inherit notions of computability from $P$. In this way we can speak of degrees contained in $X$. In particular, for the subspace $(N \rightarrow N)$ of $P$ we get the usual Turing degrees.

(6) In order to pass to higher types, it is best to consider quotients of subspaces of $P$. The simplest way to do this is to use partial equivalence relations (PERs) on $P$. The PERs form a cartesian closed category, and all the PERs inherit both topology and computability from $P$. WARNING: The PERs are not in themselves topologically defined, being arbitrary quotients, nor does topology determine the function spaces in general. EXAMPLE: Let $N$ be the identity relation on the (singletons of the) integers. Then the types as PERs: $(\ldots((N \rightarrow N) \rightarrow N) \ldots \rightarrow N) \rightarrow N$ give us the well known Kleene-Kreisel Countable Functionals.

(7) Operations on PERs can be expanded to include products and sums of dependent type systems. Invoking the propositions-as-types paradigm gives us a modeling of Martin-Löf Type Theory which additionally has computability as a modal operator ($\#$).
(8) By combining Type Theory and Computability Theory, we have a way of doing (much more than) Bishop-style Constructive Analysis in a definite model universe — rather than as a formal (intuitionistic) theory. **WARNING:** The topological reals $\mathbb{R}_t$ as a subspace of $P$ is OK, but it is better to use the Cauchy reals $\mathbb{R}_c$ as Bishop does. We can map $\mathbb{R}_c$ onto $\mathbb{R}_t$, but there is no inverse mapping, since there is no way to pick a Cauchy sequence for each real in a continuous way. Dag Normann has found it a hard study to understand the higher types $((\ldots((\mathbb{R}_c \rightarrow \mathbb{R}_c) \rightarrow \mathbb{R}_c) \ldots \rightarrow \mathbb{R}_c) \rightarrow \mathbb{R}_c)$.

(9) The simple and natural embedding of familiar structures as types over $P$ ought to allow us to incorporate much past work about computable structures (e.g., algebras) into work in this model. And we can work easily with higher types.

(10) Recursive isomorphism of types ought to lead to new problems in a new "cardinal arithmetic". Older work of Myhill, Nerode & Co. should be directly relevant.

(11) Other notions of computability can be introduced by going further up the Kleene Hierarchy — and even into the Constructible Hierarchy as done in Admissible Set Theory. The underlying universe $P$ has only the cardinality of the continuum, but it is a very rich universe.

(12) By using intersections of types and type polymorphism, it is possible that many ideas of general computability could be defined with a quite limited suite of logical notions.

(13) Even though the facts were presented semantically via a special kind of modeling on $P$, once some general principles are established, further proofs can be done formally in the style of Bishop Constructive Analysis or of Martin-Löf Type Theory.

A FINAL WARNING: There is all too often a long gap between an initial definition and a complete analysis of a concept. But simple, well structured definitions in a uniform framework ought to make the bridging of gaps easier and more understandable.