The λ-Calculus & Enumeration Operators

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What is λ-Calculus?

The calculus gives rules for the ***explicit definition*** of functions; however, the ***type-free*** version also permits ***recursion*** and ***self-replication.***

α-conversion

λX.[...X...] = λY.[...Y...]

β-conversion

(λX.[...X...])(T) = [...T...]

η-conversion

λX.F(X) = F

Church's original system (1932) also had rules for ***logic***, but that was the system proved ***inconsistent*** by Kleene-Rosser in (1936).

The names of the rules are due to Curry.
The last rule fails in many interpretations, and special efforts are needed to make it valid.

The Easy λ-Calculus Model

$$(n,m) = 2^n(2m+1)$$

$$\mathbf{set}(0) = \varnothing$$

$$\mathbf{set}((n,m)) = \mathbf{set}(n) \cup \{m\}$$

$$X^* = \{n \mid \mathbf{set}(n) \subseteq X\}$$

Application

$$F(X) = \{ m \mid \exists n \in X^*.\ (n,m) \in F \}$$

Abstraction

$$\lambda X.[\ldots X\ldots] =$$

$$\{0\} \cup \{(n,m) \mid m \in [\ldots \mathbf{set}(n) \ldots]\}$$

Every set F of integers can be used as above as an enumeration operator. The operator is computable if the set is r.e. Many compound contexts do define enumeration operators, but some precise definitions have to be given to explain which are appropriate.

Defining Continuous Operators

Definition. An operator, $\Phi(X_0, X_1, \ldots, X_{n-1})$, from sets of integers to sets of integers is called ***monotone*** iff for all values of the variables, whenever $X_i \subseteq Y_i$ for all $i<n$, we then have $\Phi(X_0, X_1, \ldots, X_{n-1}) \subseteq \Phi(Y_0, Y_1, \ldots, Y_{n-1})$.

Definition. A monotone operator Φ is ***continuous*** iff whenever $m \in \Phi(X_0, X_1, \ldots, X_{n-1})$, then there are $x_i \in X_i{*}$ for all $i<n$, such that $m \in \Phi(\mathbf{set}(x_0), \ldots, \mathbf{set}(x_{n-1}))$.

In other words: any finite amount of information about an output is already determined by some finite amount of information about the inputs.

Theorem. The continuous operators are exactly the same as the ***topologically continuous*** functions on the ***powerset*** of the set N of integers using the sets $\mathcal{U}_n = \{X \mid n \in X{*}\}$ as a basis for the topological ***neighborhoods***.

Some Continuous Operators

Theorem. The ***application operation***, $F(X)$, is continuous.

Theorem. If the operator $\Phi(X_0, X_1, \ldots, X_{n-1})$ is continuous, then so is the ***abstraction*** $\lambda X_0 . \Phi(X_0, X_1, \ldots, X_{n-1})$.

Theorem. The ***composition*** of continuous operators is always continuous.

Corollary. All ***λ-terms*** define continuous operators — even if they contain ***constants***.

Theorem. All these operators are continuous:

$$X \cap Y$$

$$X \cup Y$$

$$X^*$$

$$X + Y = \{n+m \mid n \in X \ \& \ m \in Y\}$$

$$X \times Y = \{(n, m) \mid n \in X \ \& \ m \in Y\}$$

$$X \oplus Y = (\{0\} \times X) \cup (\{1\} \times Y)$$

$$\mathbf{test}(X, Y, Z) = \{m \in Y \mid 0 \in X\} \cup \{m \in Z \mid \exists n . n+1 \in X\}$$

$$\mathbf{succ}(X) = \{n+1 \mid n \in X\}$$

$$\mathbf{pred}(X) = \{n \mid n+1 \in X\}$$

Validating Axioms

Theorem. If $\Phi(X)$ is continuous, then $\lambda X.\Phi(X)$ is the ***largest*** set F such that for all sets T, $F(T) = \Phi(T)$.

Corollary. The λ-***terms*** as interpreted in the model do indeed satisfy ***α-conversion*** and ***β-conversion***.

Corollary. For all sets F we have
$F \subseteq \lambda X.F(X)$.

Corollary. If $F = \lambda X.\Phi(X)$, then

$$F = \lambda X.F(X).$$

Theorem. If both operators $\Phi(X)$ and $\Psi(X)$ are continuous, then

$$\lambda X.\Phi(X) \subseteq \lambda X.\Psi(X) \text{ iff } \forall X.\Phi(X) \subseteq \Psi(X)$$

and

$$\lambda X.(\Phi(X) \cup \Psi(X)) = \lambda X.\Phi(X) \cup \lambda X.\Psi(X).$$

Least Fixed Points

Theorem. If $\Phi(X)$ is continuous, then Φ has a ***least fixed point*** $P = \Phi(P)$ given by

$$P = \bigcup_n \Phi^n(\varnothing).$$

Theorem. If $\Phi(X)$ is continuous and if

$\nabla = \lambda X.\Phi(X(X))$, then $P = \nabla(\nabla)$ ***is*** the

least fixed point of Φ.

Definition. The following are called the ***arithmetic combinators***:

$$\mathbf{K} = \lambda X.\lambda Y.X$$

$$\mathbf{S} = \lambda X.\lambda Y.\lambda Z.X(Z)(Y(Z))$$

$$\mathbf{Test} = \lambda X.\lambda Y.\lambda Z.\mathbf{test}(X,Y,Z)$$

$$\mathbf{Succ} = \lambda X.\mathbf{succ}(X)$$

$$\mathbf{Pred} = \lambda X.\mathbf{pred}(X)$$

Theorem. In the model the arithmetic combinators all denote ***recursive sets***, and the ***recursively enumerable sets*** are their combinations using application alone.

A Single Generator

Definition.

$$\mathbf{G} = (\lambda x.\ \mathbf{Test}(x)(\mathbf{H}_0)(\{0\}))\backslash\{0\}$$

$$\mathbf{H}_0 = \lambda x.\ \mathbf{Test}(x)(\mathbf{K})(\mathbf{H}_1)$$

$$\mathbf{H}_1 = \lambda x.\ \mathbf{Test}(x)(\mathbf{S})(\mathbf{H}_2)$$

$$\mathbf{H}_2 = \lambda x.\ \mathbf{Test}(x)(\mathbf{Test})(\mathbf{H}_3)$$

$$\mathbf{H}_3 = \lambda x.\ \mathbf{Test}(x)(\mathbf{Succ})(\mathbf{Pred})$$

Theorem. In the model the combinator **G** generates all, and only, the ***recursively enumerable sets*** under application alone.

Theorem. The semilattice of finitely generated applicative subalgebras of the model containing **G** is isomorphic to the semilattice of **enumeration degrees.**

Question. Are there applicative **G**-subalgebras requiring ***infinitely many*** (even ***uncountably many***) generators?

A Universal Space

All spaces here will be T_0-spaces.

Definition. Let $\mathbb{P}$ be the ***powerset*** of the integers $\mathbb{N}$ with the $\mathcal{U}_n = \{ X \mid n \in X^* \}$ as a ***basis*** for the topology. Additionally use the convention that each n = {n}, so that $\mathbb{N} \subseteq \mathbb{P}$ with a ***discrete*** subspace topology.

Theorem. Every ***countably based*** T_0-space $\mathbb{X}$ is ***homeomorphic*** to a ***subspace*** of $\mathbb{P}$.

Theorem. Every continuous function $\Phi : \mathbb{X} \longrightarrow \mathbb{P}$ on a ***subspace*** $\mathbb{X} \subseteq \mathbb{Y}$ can be ***extended*** to a ***maximal*** continuous $\Phi^* : \mathbb{Y} \longrightarrow \mathbb{P}$ on the ***superspace.***

The space $\mathbb{P}$ is an example of an injective T_0-space. Such spaces form a very extensive category.

The Category of Retractions

Definition. $\mathbf{D} \in \mathbb{P}$ is a ***retraction*** iff

$$\mathbf{D} = \lambda x.\, \mathbf{D}(\mathbf{D}(x)) \subseteq \lambda x.\, x$$

For two retractions $\mathbf{D}$ and $\mathbf{E}$ write

$$F: \mathbf{D} \to \mathbf{E} \text{ iff } F = \lambda x.\, \mathbf{E}(F(\mathbf{D}(x))) \text{ and let}$$

$$(\mathbf{D} \to \mathbf{E}) = \lambda F.\, \lambda x.\, \mathbf{E}(F(\mathbf{D}(x))).$$

Theorem. The category of ***retractions*** of $\mathbb{P}$ is isomorphic to the topological category of ***injective subspaces*** of $\mathbb{P}$.

Definition. $x, y \in \mathbb{P}$ write

$$\mathbf{Pair}(x)(y) = \{2n \mid n \in x\} \cup \{2m+1 \mid m \in y\},$$

$$\mathbf{Fst}(z) = \{n \mid 2n \in z\},$$

$$\mathbf{Snd}(z) = \{m \mid 2m+1 \in z\}.$$

For two retractions $\mathbf{D}$ and $\mathbf{E}$ write

$$(\mathbf{D} \times \mathbf{E}) = \lambda z.\, \mathbf{Pair}(\mathbf{D}(\mathbf{Fst}(z)))(\mathbf{E}(\mathbf{Snd}(z))).$$

Theorem. The category of ***retractions*** of $\mathbb{P}$ is a ***cartesian closed category.***

Remember: Cartesian closed categories are the models of typed λ-calculus.