Dependent Types & λ-Calculus

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What is a Type?

Definition. Recall that for $X, Y \in \mathbb{P}$ we write

$$\textbf{Pair}(X)(Y) = (X, Y) = \{2n \mid n \in X\} \cup \{2m+1 \mid m \in Y\},$$

$$\textbf{Fst}(Z) = \{n \mid 2n \in Z\}, \text{ and}$$

$$\textbf{Snd}(Z) = \{m \mid 2m+1 \in Z\}.$$

So we regard $\mathbb{P} = \mathbb{P} \times \mathbb{P}$, and for $\mathcal{A} \subseteq \mathbb{P}$ we write

$$X \mathcal{A} Y \text{ iff } (X, Y) \in \mathcal{A}.$$

Definition. By a ***type*** over $\mathbb{P}$ we understand a ***partial equivalence relation*** $\mathcal{A} \subseteq \mathbb{P}$ where, for all $X, Y, Z \in \mathbb{P}$, we have

$X \mathcal{A} Y$ implies $Y \mathcal{A} X$, and

$X \mathcal{A} Y$ and $Y \mathcal{A} Z$ imply $X \mathcal{A} Z$.

Additionally we write $X : \mathcal{A}$ iff $X \mathcal{A} X$.

Note: It is better NOT to pass to equivalence classes and the quotient spaces. But we can THINK in those terms if we like.

Definition. For subspaces $\mathbb{X} \subseteq \mathbb{P}$ write

$$[\mathbb{X}] = \{(X, X) \mid X \in \mathbb{X}\},$$

so that we may regard ***subspaces*** as ***types***.

The Category of Types

Definition. The ***product*** of types $\mathcal{A},\mathcal{B} \subseteq \mathbb{P}$ is defined as that relation where $X(\mathcal{A}\times\mathcal{B})Y$ iff $\mathbf{Fst}(X)\,\mathcal{A}\,\mathbf{Fst}(Y)$ and $\mathbf{Snd}(X)\,\mathcal{B}\,\mathbf{Snd}(Y)$.

Theorem. The product of two types is again a type, and we have
$X{:}(\mathcal{A}\times\mathcal{B})$ iff $\mathbf{Fst}(X){:}\mathcal{A}$ and $\mathbf{Snd}(X){:}\mathcal{B}$

Definition. The ***exponentiation*** of types $\mathcal{A},\mathcal{B} \subseteq \mathbb{P}$ is that relation where $F(\mathcal{A}\rightarrow\mathcal{B})G$ iff

$\forall X,Y.\ X\,\mathcal{A}\,Y$ implies $F(X)\,\mathcal{B}\,G(Y)$.

Theorem. The exponentiation (= function space) of two types is again a type, and we have
$F{:}\mathcal{A}\rightarrow\mathcal{B}$ iff $\forall X.\ X{:}\mathcal{A}$ implies $F(X){:}\mathcal{B}$.

Note: **Types do form a category — expanding the topological category of subspaces — but we wish to prove much, much more.**

Isomorphism of Types

Definition. The ***sum*** of types $\mathcal{A},\mathcal{B} \subseteq \mathbb{P}$ is defined as that relation where $X(\mathcal{A}+\mathcal{B})Y$ iff

either $\exists X_0,Y_0[X_0\mathcal{A}Y_0 \ \& \ X=(0,X_0) \ \& \ Y=(0,Y_0)]$

or $\exists X_1,Y_1[X_1\mathcal{B}Y_1 \ \& \ X=(1,X_1) \ \& \ Y=(1,Y_1)]$.

Theorem. The sum of two types is again a type, and we have

$X:(\mathcal{A}+\mathcal{B})$ iff either $\mathbf{Fst}(X)=0 \ \& \ \mathbf{Snd}(X):\mathcal{A}$

or $\mathbf{Fst}(X)=1 \ \& \ \mathbf{Snd}(X):\mathcal{B}$.

Definition. Two types $\mathcal{A},\mathcal{B} \subseteq \mathbb{P}$ are ***isomorphic***, in symbols $\mathcal{A}\cong\mathcal{B}$, provided there are

$F:\mathcal{A}\to\mathcal{B}$ and $G:\mathcal{B}\to\mathcal{A}$ where

$\forall X:\mathcal{A}.\ X\ \mathcal{A}\ G(F(X))$ and $\forall Y:\mathcal{B}.\ Y\ \mathcal{B}\ F(G(Y))$.

Theorem. If types $\mathcal{A}_0\cong\mathcal{B}_0$ and $\mathcal{A}_1\cong\mathcal{B}_1$, then

$(\mathcal{A}_0\times\mathcal{A}_1)\cong(\mathcal{B}_0\times\mathcal{B}_1)$, and

$(\mathcal{A}_0+\mathcal{A}_1)\cong(\mathcal{B}_0+\mathcal{B}_1)$, and

$(\mathcal{A}_0\to\mathcal{A}_1)\cong(\mathcal{B}_0\to\mathcal{B}_1)$.

Some Categorical Properties

Definition. Let $\mathcal{T}$ be the class of all types on the powerset space $\mathbb{P}$.

Theorem. Isomorphism is is an ***equivalence relation*** on $\mathcal{T}$, and for all $\mathcal{A},\mathcal{B},\mathcal{C} \in \mathcal{T}$,

$(\mathcal{A}\times\mathcal{B}) \cong (\mathcal{B}\times\mathcal{A})$, and $(\mathcal{A}+\mathcal{B}) \cong (\mathcal{B}+\mathcal{A})$, and

$((\mathcal{A}\times\mathcal{B})\times\mathcal{C}) \cong (\mathcal{A}\times(\mathcal{B}\times\mathcal{C}))$, and

$((\mathcal{A}+\mathcal{B})+\mathcal{C}) \cong (\mathcal{A}+(\mathcal{B}+\mathcal{C}))$, and

$(\mathcal{A}\times(\mathcal{B}+\mathcal{C})) \cong (\mathcal{A}\times\mathcal{B})+(\mathcal{A}\times\mathcal{C})$, and

$((\mathcal{A}\times\mathcal{B})\to\mathcal{C}) \cong (\mathcal{A}\to(\mathcal{B}\to\mathcal{C}))$, and

$(\mathcal{A}\to(\mathcal{B}\times\mathcal{C}) \cong (\mathcal{A}\to\mathcal{B})\times(\mathcal{A}\to\mathcal{C}))$, and

$((\mathcal{A}+\mathcal{B})\to\mathcal{C}) \cong (\mathcal{A}\to\mathcal{C})\times(\mathcal{B}\to\mathcal{C}))$.

Theorem. The types on the powerset space $\mathbb{P}$ form a ***bi-cartesian closed category***, and the isomorphism classes of types satisfy all the usual laws of ***addition***, ***multiplication***, and ***exponentiation***.

Dependent Types

Definition. Given $\mathcal{A} \in \mathcal{T}$, an ***$\mathcal{A}$-indexed family of types*** is a function $\mathcal{B}:\mathbb{P} \to \mathcal{T}$, such that

$\forall X_0, X_1.\ X_0\ \mathcal{A}\ X_1$ implies $\mathcal{B}(X_0) = \mathcal{B}(X_1)$.

Definition. The ***dependent product*** of an ***$\mathcal{A}$-indexed family of types***, $\mathcal{B}$, is defined as that relation such that

$F_0(\prod X:\mathcal{A}.\mathcal{B}(X))F_1$ iff

$\forall X_0, X_1.\ X_0\ \mathcal{A}\ X_1$ implies $F_0(X_0)\ \mathcal{B}(X_0)\ F_1(X_1)$.

Definition. The ***dependent sum*** of an ***$\mathcal{A}$-indexed family of types***, $\mathcal{B}$, is defined as that relation such that

$Z_0(\sum X:\mathcal{A}.\mathcal{B}(X))Z_1$ iff

$\exists X_0, Y_0, X_1, Y_1 [X_0 \mathcal{A} X_1\ \&\ Y_0 \mathcal{B}(X_0) Y_1\ \&$

$Z_0 = (X_0, Y_0)\ \&\ Z_1 = (X_1, Y_1)]$

Theorem. The dependent products and dependent sums of indexed families of types are again types.

Systems of Dependent Types

Definition. We say that $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ form ***a system of dependent types*** iff

- $\forall X_0, X_1. [X_0 \mathcal{A} X_1 \Rightarrow \mathcal{B}(X_0) = \mathcal{B}(X_1)]$, and
- $\forall X_0, X_1, Y_0, Y_1. [X_0 \mathcal{A} X_1 \ \& \ Y_0 \mathcal{B}(X_0) Y_1 \Rightarrow \mathcal{C}(X_0, Y_0) = \mathcal{C}(X_1, Y_1)]$, and
- $\forall X_0, X_1, Y_0, Y_1, Z_0, Z_1. [X_0 \mathcal{A} X_1 \ \& \ Y_0 \mathcal{B}(X_0) Y_1 \ \& \ Z_0 \mathcal{C}(X_0, Y_0) Z_1 \Rightarrow \mathcal{D}(X_0, Y_0, Z_0) = \mathcal{D}(X_1, Y_1, Z_1)]$,

provided that $\mathcal{A} \in \mathcal{T}$, and $\mathcal{B}, \mathcal{C}, \mathcal{D}$ are functions on $\mathbb{P}$ to $\mathcal{T}$ of the indicated number of arguments.

Note: **Clearly the definition can be extended to systems of any number of terms.**

Theorem. Under the above assumptions on $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$, we always have

$$\prod X{:}\mathcal{A}. \sum Y{:}\mathcal{B}(X). \prod Z{:}\mathcal{C}(X, Y).\ \mathcal{D}(X, Y, Z) \in \mathcal{T}.$$

Asserting Propositions

Definition. Every type $\mathcal{P} \in \mathcal{T}$ can be regarded as a ***proposition*** where ***asserting*** (or ***proving*** $\mathcal{P}$) means finding ***evidence*** $\mathtt{E}:\mathcal{P}$.

Note: Under this interpretation of logic,
asserting $(\mathcal{P} \times \mathcal{Q})$ means asserting a conjunction,
asserting $(\mathcal{P} + \mathcal{Q})$ means asserting a disjunction,
asserting $(\mathcal{P} \to \mathcal{Q})$ means asserting an implication,
asserting $(\prod \mathtt{X}:\mathcal{A}.\mathcal{P}(\mathtt{X}))$ means asserting a universal quantification, and
asserting $(\sum \mathtt{X}:\mathcal{A}.\mathcal{B}(\mathtt{X}))$ means asserting an existential quantification.

Definition. For $\mathcal{A} \in \mathcal{T}$ the ***identity type*** on $\mathcal{A}$ is defined as that relation such that $\mathtt{Z}(\mathtt{X} \equiv_{\mathcal{A}} \mathtt{Y})\mathtt{W}$ iff $\mathtt{Z}\ \mathcal{A}\ \mathtt{X}\ \mathcal{A}\ \mathtt{Y}\ \mathcal{A}\ \mathtt{W}$.

Example: Given $\mathtt{F}:(\mathcal{A} \to (\mathcal{A} \to \mathcal{A}))$, then asserting

$$\textstyle\prod \mathtt{X}:\mathcal{A}.\prod \mathtt{Y}:\mathcal{A}.\prod \mathtt{Z}:\mathcal{A}.\ \mathtt{F}(\mathtt{X})(\mathtt{F}(\mathtt{Y})(\mathtt{Z})) \equiv_{\mathcal{A}} \mathtt{F}(\mathtt{F}(\mathtt{X})(\mathtt{Y}))(\mathtt{Z})$$

is the same as asserting that $\mathtt{F}$ is an associative operation.

Computability as a Modality

Definition. Let $\Sigma \subseteq \mathbb{P}$ be the set of all the ***recursively enumerable*** subsets of $\mathbb{N}$. For $\mathcal{A} \in \mathcal{T}$ define $\#\mathcal{A} = \mathcal{A} \cap \Sigma$.

Note: Because $\mathbb{P} = \mathbb{P} \times \mathbb{P}$ and $\Sigma = \Sigma \times \Sigma$, they are trivially isomorphic types. But they should not be confused with the non-isomorphic types $[\mathbb{P}]$ and $[\Sigma]$.

Theorem. $([\mathbb{N}] \to [\mathbb{N}])$, $([\mathbb{N}] \to [\mathbb{N} \cup \{\varnothing\}])$, $\#([\mathbb{N}] \to [\mathbb{N}])$, and $\#([\mathbb{N}] \to [\mathbb{N} \cup \{\varnothing\}])$ correspond to the ***Baire space***, the superspace of ***partial functions***, the space of **total recursive functions**, and the space of **partial recursive functions.**

Example: For $\mathcal{A}, \mathcal{B} \in \mathcal{T}$, asserting

$$\# \textstyle\sum F{:}\mathcal{A} \to \mathcal{B}. \sum G{:}\mathcal{B} \to \mathcal{A}.$$

$$(\textstyle\prod X{:}\mathcal{A}. X \equiv_{\mathcal{A}} G(F(X)) \times \prod Y{:}\mathcal{B}. Y \equiv_{\mathcal{B}} F(G(Y)))$$

is the same as asserting that $\mathcal{A}$ and $\mathcal{B}$ are recursively isomorphic.

Note: $\mathcal{T}$ is also a complete lattice, being closed under arbitrary intersection.

Combining Logic & Computability

The types over $\mathbb{P}$ model Martin-Löf Type Theory, including the usual dependent types, the polymorphic types (via $\bigcap$), and computable notions — both for elements and for mappings. Via #, computability can be required as needed.

Logical validities, then, are those propositions asserted by computable elements in Σ.

The modality # also has a dual notion a making the assertion of $(\#\mathcal{A} \to \mathcal{B})$ equivalent to asserting $(\mathcal{A} \to \flat\mathcal{B})$.

Further refinements could be introduced by using the modalities of other applicative subalgebras of $\mathbb{P}$, such as $\Sigma^0{}_n$ or $\Sigma^1{}_n$.