Mixing Modality and Probability

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Logic Colloquium, September 2009

My Main Question

What are ***individuals*** and ***structures*** in Modal Logic?

A Possible Answer

Lattice-valued models give a wealth of examples of naturally defined types with well structured ***individuals*** and ***operations*** and ***relations*** on them.

Experience with lattices (and sheaves) can then suggest further ***generalizations***.

In fact, we can extend the modeling to a ***modal ZF*** where every formula has a ***probability***.

What is a Lewis (S4) Algebra?

A complete Boolean algebra plus a "necessity" operator satisfying:

$$\Box 1 = 1$$

$$\Box\Box p = \Box p \leq p$$

$$\Box(p \wedge q) = \Box p \wedge \Box q$$

Note: The second two laws can be combined:

$$\Box p = \bigvee\{q \mid q = \Box q \leq p\}.$$

"Possibility" is defined as $\Diamond p = 1 - \Box(1 - p)$.

Some Abbreviations

Ha = Heyting Algebra

cHa = Complete Heyting Algebra

Ba = Boolean Algebra

cBa = Complete Boolean Algebra

La = Lewis Algebra

cLa = Complete Lewis Algebra

What is a Frame?

Definition. A ***frame*** is complete lattice which is $(\wedge\bigvee)$-***distributive***.

Theorem. In a cLa the $\Box$-stable elements form a ***subframe***.

Theorem. In a cBa ***any*** subframe creates a cLa.

We can define: $\Box p = \bigvee\{q \in \mathbf{H} \mid q \leq p\}$, where **H** is the subframe. Such structures can be regarded as **abstract** topological spaces.

An Important Theorem

Theorem. ***Every*** frame can be made into a cHa.

Hint: $q \rightarrow r = \bigvee\{p \mid p \wedge q \leq r\}$.

Corollary. In a cHa every subframe can be regarded as a cHa (but ***not*** with the same $\rightarrow$).

Whence comes the topological interpretation of intuitionistic logic (Tarski/Stone).

Boole vs. Heyting vs. Lewis

Theorem. For every cBa **B**, there is an *interesting* cHa **H** such that $\mathbf{B} = \{\neg\neg p \mid p \in \mathbf{H}\}$.

Note: $\neg p = p \rightarrow 0$

Theorem. For every cHa **H**, there is a (non-canonical) cLa **L** such that

$\mathbf{H} = \{\Box p \mid p \in \mathbf{L}\}$.

Note: **In the category of frames, the first is a quotient and the second a subframe.**

First-Order Algebraic Semantics

$$[\![aRb]\!] = \text{given}$$

$$[\![\Phi \wedge \Psi]\!] = [\![\Phi]\!] \wedge [\![\Psi]\!]$$

$$[\![\Phi \vee \Psi]\!] = [\![\Phi]\!] \vee [\![\Psi]\!]$$

$$[\![\Phi \rightarrow \Psi]\!] = [\![\Phi]\!] \rightarrow [\![\Psi]\!]$$

$$[\![\Box\Phi]\!] = \Box[\![\Phi]\!]$$

$$[\![\exists x.\Phi(x)]\!] = \bigvee_{a \in A} [\![\Phi(a)]\!]$$

$$[\![\forall x.\Phi(x)]\!] = \bigwedge_{a \in A} [\![\Phi(a)]\!]$$

Topology vs. Probability

Proposition. For every topological space **X**, the ***powerset*** P(**X**) is a cBa, and the ***lattice of open subsets*** Op(**X**) is a cHa and a ***subframe***.

Note: These examples include the Kripke models.

Theorem. For the standard ***probability space*** Borel(**[0, 1]**) with ***Lebesgue measure***, the ***measure algebra*** Borel(**[0, 1]**)/Null is a cBa, and the quotient Op(**[0, 1]**)/Null is a cHa and is a ***proper subframe***.

Note: Call this cLa M. Think of it as a pointless space. For p ∈ M write |p| for the measure of p.

Structure of the Measure Algebra

$G_\delta = M = F_\sigma$ measurable

□p=p	G open	F closed	◊p=p
□◊p=p	□F reg. open	◊G reg. closed	◊□p=p
□p=◊p=p	G∩F = C clopen	Boolean but uncountable	
|p|∈ℚ	Q rational	not Boolean	
$G = B_\sigma$	B basic	countable Boolean from intervals with rational ends	

Note: Using the measure algebra, every modal logical formula has a probability. Owing to the continuous automorphisms of M, every pure statement without free variables has truth value either 0 or 1.

Proving M ≠ G ≠ □F

Theorem. There is a construction similar to that of the Cantor Discontinuum with U∪K = [0, 1] with U dense, open, K closed, nowhere dense, and of positive measure, and U∩K = ∅.

Corollary 1. Let k = K/Null. Then k ∈ **M** and k ∉ **G**.

Proof: Suppose k ∈ **G**. Let V be an open set, V/Null = K/Null, giving V-K∈Null. But, this difference is open and so is ∅. So V⊆K, which implies V = ∅. But, K-V∈Null and K∉Null.

Corollary 2. Let u = U/Null. Then u ∈ **G** and u ∉ □**F**.

Proof: Suppose u ∈ □**F**. Then u = □(1-w), with w ∈ **G**, a negation in the cHa **G**. By the laws of triple negation u = □◊u. But U is dense, so ◊u = 1 and hence u = 1. This would imply k = 0, which is false.

Proof of the 0-1 Law

Theorem. There is no 0 < p < 1 invariant under the group Γ of all continuous, measure-preserving automorphisms of M.

Lemma. If a,b ∈ **B**, |a| = |b|, then there is a τ ∈ Γ with τ(a) = b.

Proof of the Theorem:

- Find g ∈ **G** with p ≤ g < 1, and so 1-g > 0.
- Find b ∈ **B** with b ≤ g, b ∧ p > 0, and |b| ≤ |1-g|.
- Find h ∈ **G** with 1-g ≤ h, and |h ∧ g| < |b ∧ p|.
- Find a ∈ **B** with a ≤ h, and |a| = |b|.

Now a ∧ p ≤ h ∧ g, and so |a ∧ p| < |b ∧ p|.

But let τ(a) = b and so τ(a ∧ p) = b ∧ p. **Contradiction!**

Extension vs. Intension

$\exists y\ [x = y \wedge \Phi(y)]$ *vs.* $\Box\Phi(x)$

Different principles hold in different contexts:

$$\sigma = \tau \wedge \Phi(\sigma) \rightarrow \Phi(\tau)$$

vs.

$$\Box[\sigma = \tau] \wedge \Phi(\sigma) \rightarrow \Phi(\tau)$$

The prime example of an intensional mapping:

$$\Box[\![\Phi \leftrightarrow \Psi]\!] \leq [\![\Box\Phi \leftrightarrow \Box\Psi]\!]$$

What is an L-Set?

Definition. An **L**-set is a set A equipped with an **L**-valued equality $[\![x = y]\!]$, where for all $x,y,z \in A$

$$[\![x = x]\!] = 1\ ;$$
$$[\![x = y]\!] = [\![y = x]\!];\ \text{and}$$
$$[\![x = y]\!] \wedge [\![y = z]\!] \leq [\![x = z]\!].$$

Note: There is a useful notion of complete L-set and a process of completion.

Note: Mappings between L-sets can be either extensional or intensional.

L as an L-Set

Definition. Make **L** into an **L**-set by defining the **L**-valued equality $[\![p = q]\!] = p \leftrightarrow q$ for all $p,q \in$ **L**.

Questions: (1) Are there other interesting intensional mappings on M other than $\Box$ and $\Diamond$?

(2) Can they be used for modeling other known modal logics?

Boolean-valued Reals

Theorem. The set $\mathbb{R}_L = \{\alpha \mid \alpha : Op(\mathbb{R}) \rightarrow_{frm} \mathbf{L}\}$ can be made into a complete L-set by defining;

$$[\![\alpha = \beta]\!] = \bigwedge_{U \in Op(\mathbb{R})} (\alpha(U) \leftrightarrow \beta(U)).$$

Theorem. The frame $Op(\mathbb{R}\times\mathbb{R})$ is the ***frame-coproduct*** of $Op(\mathbb{R})$ with itself.

Theorem. Using $+ : \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}$ and $(+) : Op(\mathbb{R}) \rightarrow_{frm} Op(\mathbb{R}\times\mathbb{R})$, then for $\alpha,\beta : Op(\mathbb{R}) \rightarrow_{frm} \mathbf{L}$ we have $(\alpha,\beta) : Op(\mathbb{R}\times\mathbb{R}) \rightarrow_{frm} \mathbf{L}$, and so we can define $(\alpha+\beta) = (\alpha,\beta)\circ(+) : Op(\mathbb{R}) \rightarrow_{frm} \mathbf{L}$.

Note: Other continuous functions can be handled in the same way. Many laws of algebra then follow automatically.

Random Variables as Reals

Theorem. For the cLa **M** we can identify

$$\mathbb{R}_{\mathbf{M}} = \{\,(f)/\mathrm{Null} \mid (f) : \mathrm{Op}(\mathbb{R}) \to_{\sigma\text{-frm}} \mathrm{Borel}(\mathbf{[0,1]})\,\}$$

where the f: $\mathbf{[0, 1]} \to \mathbb{R}$ are measurable functions and (f) means inverse image; moreover, we can set:

$$[\![(f)/\mathrm{Null} = (g)/\mathrm{Null}]\!] = \{\, t \in \mathbf{[0,1]} \mid f(t) = g(t)\,\}/\mathrm{Null}.$$

In this representation we find:

$$(f)/\mathrm{Null} + (g)/\mathrm{Null} = (f + g)/\mathrm{Null}.$$

Note: We can similarly treat other measurable operations on the **M**-valued reals.

Intensional Powersets

Definition: Given a complete **L**-set A the ***intensional powerset*** of A is the collection of P: A→ **L** where, for all x,y∈A, we have $P(x) \wedge \Box[\![x = y]\!] \leq P(y)$.

And we use the definition

$$[\![P = Q]\!] = \bigwedge_{x\in A}(P(x) \leftrightarrow Q(x))$$

Theorem: The intensional powerset of A is a complete **L**-set.

Note: A Principle of Comprehension follows.

Question: Should we be able to iterate this notion of powerset?

A Modal Boolean-Valued Universe

$$V^{(L)} = \{v : \mathbf{dom}\, v \to \mathbf{L} \mid \mathbf{dom}\, v \subseteq V^{(L)} \;\&\; \forall x,y \in \mathbf{dom}\, v\,[\, v(x) \wedge \Box[\![x = y]\!] \leq v(y)\,]\,\}$$

$$[\![u \in v]\!] = \bigvee\{v(y) \wedge \Box[\![u = y]\!] \mid y \in \mathbf{dom}\, v\}$$

$$[\![u = v]\!] = \bigwedge\{\, u(x) \to [\![x \in v]\!] \mid x \in \mathbf{dom}\, u\,\} \wedge \bigwedge\{\, v(y) \to [\![y \in u]\!] \mid y \in \mathbf{dom}\, v\,\}$$

The new insight:

intensional ↘ u ∈ v ↖ extensional

What is MZF?

Substitution (A number of previous lemmata are needed.)

$$\Box[u = v] \wedge \Phi(u) \to \Phi(v)$$

Extensionality & Comprehension

$$\forall u,v\,[\, u = v \leftrightarrow \forall x\,[x \in u \leftrightarrow x \in v]]$$

$$\forall u\, \exists v \Box \forall x\,[x \in v \leftrightarrow x \in u \wedge \Phi(x)]$$

Singleton

$$\forall u\, \exists v \Box \forall x\,[x \in v \leftrightarrow \Box[x = u]]$$

Leibnitz' Law

$$\forall x,y\,[\Box[x = y\,] \leftrightarrow \forall u\,[x \in u \to y \in u]]$$

Definable Modality

$$\{\varnothing\} = \{\varnothing \mid \Phi\,\} \leftrightarrow \Phi$$

$$\Box\, \Phi \leftrightarrow \forall u\,[\{\varnothing\} \in u \to \{\varnothing \mid \Phi\,\} \in u]$$

Two Membership Relations?

Extensional Membership

$u \varepsilon v \leftrightarrow \exists y\, [y \in v \wedge u = y]$

Extensional Comprehension

$\forall u\, \exists v \Box \forall x\, [x \varepsilon v \leftrightarrow x \varepsilon u \wedge \exists y\, [\, \Phi(y) \wedge x = y]]$

Extensional Singleton

$\forall u\, \exists v \Box \forall x\, [x \varepsilon v \leftrightarrow x = u]$

Extensional Leibnitz' Law

$\forall x,y\, [\, x = y \leftrightarrow \forall u\, [x \varepsilon u \rightarrow y \varepsilon u]]$

Intensional Powerset

$\forall v\, \exists w \Box \forall u\, [u \in w \leftrightarrow \Box [u \subseteq v]]$

Extensional Powerset

$\forall v\, \exists w \Box \forall u\, [u \varepsilon w \leftrightarrow u \subseteq v]$

A Refutation

Theorem. In $V^{(M)}$ the following has truth value 0:

$$\forall u,v\, [\, u = v \leftrightarrow \forall x\, [x \varepsilon u \leftrightarrow x \varepsilon v]].$$

Proof: Find $p \in \mathbf{M}$ with $0 < p < 1$ and $\Box p = 0$. **(How?)**

Let $a = \{\varnothing\}$ and $b = \{\varnothing \mid p\}$, and $u = \{a \mid p\}$ and $v = \{b \mid p\}$.

We have $[\![a = b]\!] = p$, and $[\![a \in u]\!] = p$ and $[\![a \in v]\!] = 0$.

It follows that $[\![u = v]\!] = \neg p$. We also calculate that

$[\![x \varepsilon u]\!] = [\![x = a]\!] \wedge p$ and $[\![x \varepsilon v]\!] = [\![x = b]\!] \wedge p$.

But then $[\![x \varepsilon v]\!] = [\![x = a]\!] \wedge p$ as well. From this we get:

$$[\![u = v \leftrightarrow \forall x\, [x \varepsilon u \leftrightarrow x \varepsilon v]]\!] = [\![u = v]\!] = \neg p.$$

The conclusion of the theorem then follows by the 0-1 Law for M.

Using Russell's Paradox

Theorem. For each stage $V_\alpha^{(M)}$ of the universe it is possible to find an element a of the model such that

$$[\![a = y]\!] = 0 \text{ for all } y \text{ in } V_\alpha^{(M)}.$$

Proof: Apply the ***Extensional Comprehension Principle*** to have an element a where for all x in the model:

$$[\![x \varepsilon a]\!] = [\![x \varepsilon \mathbf{V}_\alpha]\!] \wedge [\![\neg\, x \varepsilon x]\!],$$

where $\mathbf{V}_\alpha$ is the constant function 1 on $V_\alpha^{(M)}$.

Putting a for x, we have $[\![a \varepsilon \mathbf{V}_\alpha]\!] = 0$.

The desired conclusion then follows.

Another Refutation

Theorem. In $V^{(M)}$ the following has truth value 0:

$$\exists v\, \forall u\, [u \in v \leftrightarrow u = \varnothing].$$

Proof: Again, find $p \in \mathbf{M}$ with $0 < p < 1$ and $\Box p = 0$.
Suppose we had v in the model where $[\![u \in v]\!] = [\![u = \varnothing]\!]$ for all u in the model. Now v is a function with **dom** $v \subseteq V_\alpha^{(M)}$ for some stage α. Find an a with $[\![a = y]\!] = 0$ for all y in $V_\alpha^{(M)}$.
Take $u = \{a \mid \neg p\}$ which implies $[\![u = \varnothing]\!] = p$. We then have $p \leq [\![u \in \mathbf{V}_\alpha]\!] = \bigvee\{\Box[\![u = w]\!] \mid w \in V_\alpha^{(M)}\}$. But we find

$$\Box[\![u = w]\!] = \Box(\neg p \rightarrow [\![a \in w]\!]) \wedge \Box \bigwedge\{w(y) \rightarrow [\![y \in u]\!] \mid y \in \mathbf{dom}\ w\} \leq \Box p,$$

But, this is impossible.

Note: We can also refute: $\forall v\, \exists w\, \forall u\, [u \in w \leftrightarrow u \subseteq v]$.

Pairs, Products, & Relations

Definitions: In any $V^{(L)}$ the following are defined:

(i) $\{u\} = \{(u,1)\}$;
(ii) $\{u, v\} = \{(u,1),(v,1)\}$;
(iii) $(u, v) = \{\{u\}, \{u, v\}\}$; and
(iv) $a \times b = \{((x, y), a(x) \wedge b(y)) \mid x \in \mathbf{dom}\ a \wedge y \in \mathbf{dom}\ b\}$.

Theorem: In any $V^{(L)}$ we have:

(i) $\forall u,v\, [\{u\} = \{v\} \leftrightarrow \Box u = v]$;
(ii) $\forall u,v,s,t\, [\{u, v\} = \{s, t\} \leftrightarrow \Box\, [u = s \wedge v = t] \vee \Box\, [u = t \wedge v = s]]$;
(iii) $\forall u,v,s,t\, [(u, v) = (s, t) \leftrightarrow \Box\, [u = s \wedge v = t]]$; and
(iv) $\forall a,b,t\, [t \in (a \times b) \leftrightarrow \exists x,y\, [\, x \in a \wedge y \in b \wedge \Box\, t = (x, y)]]$.

Relational Comprehension

$$\forall a,b \exists w \subseteq (a \times b) \Box \forall x \in a \forall y \in b\, [(x, y) \in w \leftrightarrow \Phi(x, y)]$$

Two Sub-Universes

$$U^{(M)} = \{v : \mathbf{dom}\ v \to M \mid \mathbf{dom}\ v \subseteq U^{(M)}\ \&$$
$$\forall x,y \in \mathbf{dom}\ v\, [\, v(x) \wedge \Box [\![x = y]\!] \leq \Box v(y)\,]\, \}$$

$$W^{(M)} = \{v : \mathbf{dom}\ v \to M \mid \mathbf{dom}\ v \subseteq W^{(M)}\ \&$$
$$\forall x,y \in \mathbf{dom}\ v\, [\, v(x) \wedge [\![x = y]\!] \leq v(y)\,]\, \}$$

Note: (i) The universe $U^{(M)}$ models an **inuitionistic G-valued** set theory.
(ii) The universe $W^{(M)}$ models the usual **M**-valued, **extensional Boolean-valued** set theory.
(iii) Both universes are **definable** in the modal universe $V^{(M)}$.

Random Numbers?

Alex Simpson (Edinburgh) has argued that the quotient frame $\mathrm{Op}(\mathbf{[0,1]}) \to_{\mathrm{frm}} \mathrm{Op}(\mathbf{[0,1]})/\mathrm{Null}$ (N.B. another ***pointless space***) can be considered as satisfactorily modeling ***random reals.***

Note: This space has many **M**-valued points, and it can be taken as a subset of $\mathbb{R}_M$. In fact, we can identify

$$\mathbf{Rand_M} = \{\, (f)/\mathrm{Null} \mid f\colon \mathbf{[0, 1]} \to_{\mathrm{meas}} \mathbf{[0, 1]}\ \&$$
$$\{\, t \in \mathbb{R} \mid f(t) \in N\, \} \in \mathrm{Null}, \text{ for all } N \in \mathrm{Null}\, \}.$$

These random reals have the property of avoiding all Γ-invariant **closed** subsets of $\mathbb{R}_M$ of measure zero.

Question: Do they have other interesting modal properties? How do they compare to Solovay reals?

Random Algorithms?

We can define many ***computable*** functions $f\colon \mathbb{R} \times \mathbb{N} \to \mathbb{N}$, where in evaluating $f(\alpha,n)=m$, the α is used an an ***oracle***.

Perhaps we can say we have a ***random algorithm*** just in case

$$[\![\forall n \in \mathbb{N}\ \exists m \in \mathbb{N}\ \Box \forall \alpha \in \mathbf{Rand_M}.\ f(\alpha,n)=m]\!] = 1?$$

Applying Ergodic Theory?

Recall: In the measure-algebra model of MZF, every continuous measure-preserving automorphism of M induces an automorphism of the whole universe $V^{(M)}$.
Γ is the group of all such automorphisms.

Furstenberg's Multiple Recurrence Theorem.

Let $\tau \in \Gamma$, and let $[\![\Phi(a)]\!] \neq 0$, where $\Phi(a)$ has no other parameters. Then for all k there exists an n such that

$$[\![\Phi(a) \wedge \Phi(\tau^{n}(a)) \wedge \Phi(\tau^{2n}(a)) \wedge \Phi(\tau^{3n}(a)) \wedge \ldots \wedge \Phi(\tau^{kn}(a))]\!] \neq 0.$$

Truth by Degrees?

Comment: There are many subframes of M. For example $\mathbf{D} \subseteq \mathbf{G} \subseteq \mathbf{M}$, defined as $\mathbf{D} = \{[0, r]/\text{Null} \mid r \in \mathbb{R}\}$, is closed (in M) under arbitrary sups and infs.

The modal operator Δ defined by

$$\Delta p = \bigvee \{d \in \mathbf{D} \mid d \leq p\}$$

is, of course, stronger than $\Box$ but not intensional.

Questions: But is Δ at all interesting? Would propositions with values in D be interesting? Suggestions welcome!

Are You Ready for Multiverses?

Observation: Large cBa's usually have many subframes (= abstract topologies). Each one gives a model for MZF. And indeed one cBa may give rise to many of these. For example:

- **M** measurable
- **G** open
- **S** cylindric (using higher dimensions)
- **D** real-valued degrees
- **E** broad degrees (small, medium, large)
- **T** binary degrees (all or nothing, 0 or 1)

And we have both modal and intuitionistic versions.

The End

But, to be continued, I hope!