# CLOSED CATEGORIES AND THE THEORY OF PROOFS

The main aim of this article is to suggest a translation of the simplest concepts of category theory into the language of the (structural) theory of proofs, to use this translation to simplify the proof of some known results [1], and to obtain two new ones: the coherence theorem for canonical morphisms in (non-monoidal, non-symmetric) closed categories [2], and the solution of the problem of equality of canonical morphisms. Extensions of these results on monoidal closed, symmetric closed, and monoidal symmetric closed categories, are sketched. The decision is obtained by means of a correct and faithful translation of canonical morphisms into an expansion of the $\lambda$-language, which has the tools for a special account of «superfluous» premises of implications (the thinning rule). The expansions of the $\lambda$-language which have so far appeared in the literature have not possessed this facility.

In order to understand the proofs of some of the theorems (but not their statement), some knowledge of proof theoretic techniques is required. All the essential facts will be presented in detail in a more elementary publication [3], devoted to the technically simpler and well-known cases of Cartesian closed categories. These facts can also be found in [4], [5], and [6].

The first non-trivial application of proof theory to category theory was apparently the work of Lambek [7], where he solves the word problem for systems which he calls *residuated categories*; he constructs a Gentzen-like propositional calculus whose deductions correspond to canonical morphisms, and he proves for this calculus a theorem on cut-elimination, thus obtaining decidability. In the same article Lambek was apparently the first in the literature to investigate the concept of equivalence of proofs which was later, seemingly independently, studied in proof theory ([10] and [11]). In his next paper [8] Lambek solves the word problem for (non-symmetic) biclosed monoidal categories, and establishes the coherence theorem for them. In [9] the same method is applied to Cartesian closed categories. Kelly and MacLane [1] used Lambek's method to solve the word problem for symmetric monoidal closed categories, and established a partial coherence theorem for them.

They used a symbolism similar to that of Schütte 412]. Their results are presented for non-specialists in [13, §5].

The concept of the equivalence of proofs, as presented by Lambek, is rather cumbersome, since it uses a logic-like system with rules for introducing connectives to the left and right of arrows. A similar concept (as was shown later [14]) for systems of natural deduction [10] proved to be much more manageable: one needs only compare [15] (plus [16]) and [17].

We obtain a further simplification of the method by passing from deductions to the corresponding terms [18], [6]. This allows us to combine the advantages of natural and logistic deductions and also to simplify the topology by algebraization: deductions are two-dimensional structures, and terms are one-dimensional. On the other hand we can use the method of passing from the language of combinators to the λ-language and vice versa, as in proof theory (compare [9], §4).

The main technical difficulty compared with [7], [8] and [1] lies in the absence of a conjunction (tensor product) which permits one to «transform» multiple implications $(A_1 \supset \ldots (A_n \supset B) \ldots)$ into arrows of the form $A_1 \otimes \ldots \otimes A_n \to B$. Because of this, proofs of the preservation of equivalence become much more clumsy. However we considered it essential to give them in detail, since the corresponding point in Lambek (which we have not checked) is unclear [19, pp. 110-111, 140-144].

The author is indebted to V.P. Orevkov, who proof-read the first version of the manuscript.

We recall the definitions of a category, functor and natural transformation of functors.

*A category* $K$ consists of a class of objects Ob $K$ and a class of morphisms Mor $K$, which satisfy three conditions:

1) to each ordered pair of objects $(A, B)$ is assigned a set Hom $(A, B) \subset$ Mor $K$ (whose elements are *morphisms* from $A$ to $B$), moreover the sets Hom $(A, B)$ for distinct pairs $(A, B)$ are disjoint and their union over all $A$ and $B$ is Mor $K$; instead of $f \in$ Hom $(A, B)$ we write $f : A \to B$ or $A \xrightarrow{f} B$;

2) for all $f : A \to B$, $g : A \to B$ the composition of $f$ and $g$ is defined, which is associative:

$$f \cdot (g \cdot h) = (f \cdot g) \cdot h;$$

(3) for any $A \in$ Ob $K$ there is a unique morphism $1_A$, such that

$$1_A \cdot f = f;\ g \cdot 1_A = g \text{ for any } f : B \to A,\ g : A \to C\ .$$

The dot in the notation for composition is often left out.

*A one-place covariant functor* of a category $K$ to a category $\mathfrak{L}$ is a map $F$ from $K$ into $\mathfrak{L}$ transforming objects into objects, morphisms $f : A \to B$ into morphisms $F(A) \to F(B)$ and preserving 1 and composition

$$F(1_A) = 1_{F(A)} \; ; \qquad F(f \cdot g) = F(f) \cdot F(g) \, .$$

A two-place functor from $K$ into $\mathfrak{L}$, contravariant in the first argument and covariant in the second (the functor $K^* \times K \to \mathfrak{L}$) is a mapping $F$ which transforms pairs of objects into objects, pairs of morphisms $f : A_1 \to A$, $g : B \to B_1$ into morphisms $F(f,g) : F(A,B) \to F(A_1,B_1)$, which preserves a pair of identities and (up to a permutation of the first arguments) preserves composition

$$F(1_A, 1_B) = 1_{F(A,B)} \; ; \qquad F(f_2f_1,g_1g_2) = F(f_1,g_1)\, F(f_2,g_2)$$

for

$$A_2 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A, \qquad B \xrightarrow{g_1} B_1 \xrightarrow{g_2} B_2$$

Instead of the single condition for preserving composition we shall often use the equivalent conditions for preserving composition with identity morphism, namely

$$F(f,g) = F\,(f,1_{B_1})\, F\,(1_A,g) = F\,(1_A,g)\, F\,(f, 1_B) \; ;$$

and

$$F\,(f_2f_1,1) = F\,(f_1,1)\, F\,(f_2, 1); \qquad F\,(1,g_1g_2) = F\,(1,g_1)\, F\,(1, g_2).$$

A natural transformation of a one-place covariant functor $F$ into (a one-place covariant functor) $G$ is a family of morphisms $\Phi_A : F(A) \to G(A)$, such that for any object $A$ and any morphism $a : A \to A_1$ the diagram

$$\begin{array}{ccc} F\,(A) & \xrightarrow{\varphi A} & G\,(A) \\ {\scriptstyle F\,(a)}\downarrow & & \downarrow{\scriptstyle G\,(a)} \\ F\,(A_1) & \xrightarrow{\varphi A_1} & G\,(A_1) \end{array}$$

commutes, that is

$$G(a)\, \varphi_A = \varphi_{A_1}\, F(a).$$

Analogously, a natural transformation of a functor $F$, contravariant in the first argument and covariant in the second, to a functor $G$ of the same kind, is a family of morphisms $\varphi_{AB} : F(A,B) \to G(A,B)$ such that

$$G(a,\, b)\varphi_{A,B} = \varphi_{A_1,B_1}\, F(a,\, b) \text{ for any } a\text{: } A_1 \to A,\; b : B \to B_1.$$

We shall use the definition given in [20] of natural transformations for compositions of many place functors (which themselves may fail to be functors). The necessary particular cases of these definitions will be introduced in the appropriate place.

## § 1. Closed Categories

1.1. A closed category $\mathcal{G} = (\mathcal{G}_0, V, \text{hom}, I, i, i^{-1}, j, L)$ consists of the following elements: a category $\mathcal{G}_0$, an object $I \in \text{Ob}\ \mathcal{G}_0$, functors $V: \mathcal{G}_0 \to$ Set (the category of sets) and hom: $\mathcal{G}_0^* \times \mathcal{G}_0 \to \mathcal{G}_0$ [instead of hom $(a,b)$ we write $a \supset b$],

a natural isimorphism $i = i_A : A \to I \supset A$ $\quad (i)$

its inverse $i^{-1} = i_A^{-1} : (I \supset A) \to A$ $\quad (i^{-1})$

the natural transformations $j = j_A : I \to A \supset A$ $\quad (j)$

$$L = L^A = L^A_{BC} : (B \supset C) \to (A \supset B) \supset (A \supset C) \qquad (L)$$

The above satisfy the following conditions of coherence.

CC0 $V\ \text{hom} = \text{Hom}$

CC1 $L^A_{BB} j_B = j_{(A \supset B)}$

CC2 $(j_A \supset 1_{A \supset C}) L^A_{AC} = i_{(A \supset C)}$

CC3 $(1_{B \supset C} \supset L^A_{BD})\ L^B_{CD} = (L^A_{BC} \supset 1_{(A \supset B) \supset (A \supset D)})\ L^{A \supset B}_{(A \supset C)(A \supset D)}\ L^A_{CD}$

CC4 $(i_B \supset 1_{(I \supset C)}\ L^I_{BC} = (1_B \supset i_C)$

CC5 $V i_{A \supset A}\ \ 1_A = j_A$

The naturality of a transformation is here understood as in [20]: the naturality of $j$ is expressed by the equation

$(nj)$ $\quad (a \supset 1_A) j_A = (1_I \supset a) j_I$ for any $a : I \to A$

the naturality of $L$ on $B$, $C$ is taken in the usual way

$(n\ L1)$ $\quad ((1_A \supset b) \supset 1_{A \supset C}) L^A_{B'C} = L^A_{BC}\ (b \supset 1_C)$ for $b : B' \to B$

$(n\ L2)$ $\quad (1_{A \supset B} \supset (1_A \supset C))\ L^A_{BC} = L^A_{BC'}\ (1_B \supset 1_C))$ for $c : C \to C'$

The naturality of $L^A$ in $A$ is written

$$(n\ L3)\quad ((a \supset 1_B) \supset 1_{A\supset C})L^A_{BC} = (1_{A'\supset B} \supset (a \supset 1_C))\ L^A_{BC} \quad \text{for } a : A \to A'$$

We note that CC0, together with the fact that $V$ is a functor, gives

$$(V\ (a \supset b))f = bfa \text{ for } \quad a : A \to A',\ \ f : A \to B,\ \ b : B \to B' \quad (1)$$

We list, under the same numbers as in § 2 of chapter 1 in [2], some propositions for closed categories which we shall use later

$$(Vi_x)\ (V_f)\ 1_A = fij_a \qquad \text{for } f : (A \supset A) \to X \qquad (2)$$

$$(VL^A_{BC})f = (1_A \supset f) \qquad \text{for } f : B \to C \qquad (3)$$

$$(1_A \supset f)j_A = (Vi_{A\supset B})\ f = (f \supset 1_B)j_B \qquad \text{for } f : A \to B \qquad (4)$$

$$i_{A\supset A} = (1_I \supset i_A) \qquad (5)$$

$$i_A f = (Vi_{I\supset A})\ f \qquad \text{for } f : I \to A \qquad (6)$$

$$j_I = i_I \qquad (7)$$

$$(1_I \supset f\ ) = (f \supset 1_I) \qquad \text{for } I \to I \qquad (8)$$

In future we shall often omit the upper and lower indices, leaving the reader to confirm that the isomorphisms in question have the necessary domain and range.

We deduce two more equations

$$(f \supset 1)j_A = i_A f \qquad \text{for } f : I \to A \qquad (9)$$

In fact, from $(nj)$, (7) we have $(f \supset 1)j_A = (1 \supset f)i_I = i_A f$ by the naturality of $i$.

$$(1 \supset i_C^{-1})\ L^I_{BC} = (i_B^{-1} \supset 1) \qquad (10)$$

The proof proceeds by multiplying CC4 from the left by $i_C^{-1} \supset i_C^{-1}$ For the list $\Gamma = A_1\ ,\ ...,\ A_n$, consisting of objects or functors, we put

$$\Gamma \supset A \leftrightarrows (A_1 \supset (A_2 \supset ... \supset (A_n \supset A)\ ...));$$

when $n = 0$ this simply denotes $A$. Analogously

$$1_\Gamma \leftrightarrows 1_{A1}\ I_{A2}\ ...\ I_{An}\ ;\quad L^\Gamma \leftrightarrows L^{A1}\ L^{A2}\ ...\ L^{An}$$

Later it will be more convenient to define the closed category $\mathcal{G}$ in the style of [1] as $(\mathcal{G}_0, \text{hom}, I, i, i^{-1}, j, L, \Pi^{-1})$, where all the items except $\Pi^{-1}$ are the same as before, in particular they satisfy the axioms $(i)$, $(i^{-1})$, $(j)$, $L$, and in addition, the conditions (4) and (7) (replacing CC0 and CC5), and

$$\Pi^{-1}_{AB} : \text{Hom}\ (I, A \supset B) \to \text{Hom}\ (A, B)$$

is a (natural) transformation, the inverse of

$$\Pi = \Pi_{AB} : \text{Hom}\ (A, B) \to \text{Hom}\ (I, A \supset B) \text{ where } \Pi_{AB}\ (g) = (1_A \supset g)j_A$$

We check the equivalence of this definition and the old one.

If we have the functor $V$ we define $\Pi^{-1}\ (f) = V\ (fi_I^{-1})\ 1_I$. Then

$$\Pi\ (\Pi^{-1}\ (f)) = (1 \supset V\ (fi^{-1})\ 1)j \underset{(4)}{=} (Vi)\ (Vg)\ 1 \underset{(2)}{=} gj \underset{(7)}{=} fi^{-1}\ i = f\ ,$$

where $g = fi_I^{-1}$.

$$\Pi^{-1}\ (\Pi\ (f\ )) = V\ ((\Pi f\ )i_I^{-1})\ 1 = V\ ((1 \supset f)ji^{-1}) \underset{(1)}{=} fV\ (ji^{-1})\ 1 \underset{(2)}{=} f\ (Vi^{-1})\ (ji^{-1}j)$$

$$\underset{(7)}{=} f\ (Vi^{-1})j \underset{CC5}{=} f\ 1 = f\ .$$

Conversely, if we have $\Pi^{-1}$, we put

$$VA = \begin{cases} \text{Hom}\ (C, D), & \text{if } A = (C \supset D) \\ \text{Hom}\ (I, A), & \text{otherwise} \end{cases}$$

$$(Vf)\ g = (\Pi^{-1})^0\ (f\ \Pi^0\ (g)) \quad \text{for } f : A \to B,\ g \in VA$$

where the expression $\Pi^0$ denotes $\Pi$ if $A = C \supset D$ for some $C$, $D$ and is empty otherwise. Analogously, $\Pi^{-1}$ is present only if $B = E \supset F$ for some $E$, $F$.

The property of composition for $V$ follows from $\Pi\Pi^{-1} = 1$.

We check CC0, that is

$$\Pi^{-1}\ ((f \supset h)\ \Pi\ (g)) = hgf \quad \text{for} \quad C_1 \xrightarrow{f} C,\ \ C \xrightarrow{g} D,\ \ D \xrightarrow{h} D_1$$

$$\Pi\ (h\ g\ f) = (1 \supset hgf)j = (1 \supset h\ g)\ (1 \supset f)j \underset{(4)}{=} (1 \supset h)$$

$$(1 \supset g)\ (f \supset 1)j = (f \supset h)\ (1 \supset g)j = (f \supset h)\ \Pi\ (g)$$

We check CC5, that is

$$\Pi^{-1}\ (i_{A\supset A}\ \Pi\ (1_A)) = j_A$$

$$\Pi j_A = (1_I \supset j_A)j_I \underset{(7)}{=} (1_I \supset j_A)i_I \underset{(*)}{=} (i_{A\supset A}\ j_A = i_{A\supset A}\ (1_A \supset 1_A)j_A = i_{A\supset A}\ \Pi\ (1_A)$$

where (*) follows from the naturality of $i$.

We rewrite in terms of $\Pi$ the relations which were formulated in terms of $V.\Pi$

CC0′ $\quad (f \supset h)\ \Pi\ (g) = \Pi\ (hgf)$ $\qquad$ for $C_1 \xrightarrow{f} C,\ C \xrightarrow{g} D,\ D \xrightarrow{h} D_1$

CC5 $\quad i_{A\supset A}\ \Pi\ (1_A) = \Pi\ (j_A)$

$\Pi\ (fj_A) = i_x f j_A$ $\qquad$ for $f : A \supset A \to X$ $\qquad$ (2′)

$L^A_{BC}\ \Pi\ (f) = \Pi\ (1_A \supset f)$ $\qquad$ for $f : B \to C$ $\qquad$ (3′)

$(1_A \supset f)jA = \Pi\ (f) = \Pi^{-1}\ (i_{A\supset B}\ \Pi\ (f)) = (f \supset 1_B)j_B$ $\qquad$ for $f : A \to B$ $\qquad$ (4′)

$\Pi\ (i_A f) = i_{I\supset A}\ \Pi\ (f)$ $\qquad$ for $f : I \to A$ $\qquad$ (6′)

$\Pi\ (f) = i_A f$ $\qquad$ for $f : I \to A$ $\qquad$ (9′)

When referring to these relations we shall omit the dashes.

## 2. Hilbert-type systems HCC

We construct a deductive system HCC (a Hilbert-type system for closed categories) and we turn it into a closed category.

The atomic formulas are constant $I$ and propositional variables. The formulas are constructed from the atomic ones by means of the two-place connective hom. Sequents of an HCC system are expressions of the sort $A \to B$, where $A$, $B$ are formulas.

Deductions and deducible sequents are defined inductively. Instead of «$b$ is a deduction of the sequent $A \to B$», we write $b : A \to B$ or $A \xrightarrow{b} B$. We list (2.1.1. - 2.2.2.) the axioms and inference rules.

2.1. Postulates* of a category.

2.1.1. The identity $1_A : A \to A$.

---

* Postulates in general include both axioms and inference rules.

2.1.2. Composition. If $f : A \to B$, $g : B \to C$, then expression $(g \cdot f)$ is a deduction of $A \to C$. This deduction is written in the following way (tree-like notation)

$$\frac{A \xrightarrow{b} B,\ B \xrightarrow{g} C}{A \xrightarrow{g \cdot f} C}$$

We often omit the dots and brackets.

2.1.3. hom.

If $a : A_1 \to A$, $A$, $b : B \to B_1$, then the expression $a \supset b$ is a deduction of the sequent $A \supset B \to (A_1 \supset B_1)$. In tree notation

$$\frac{A_1 \xrightarrow{a} A \quad B \xrightarrow{b} B_1}{A \supset B \xrightarrow{a \supset b} A_1 \to B_1}$$

2.2. Postulates of a closed category.

2.2.1. Axioms $(i)$, $(i^{-1})$, $(j)$, $(L)$ from § 1.

2.2.2. The implication-elimination

$$\frac{I \xrightarrow{b} A \supset B}{A \xrightarrow{\Pi^{-1}(b)} B}$$

Derivations in HCC will also be called *combinators* (readers with even a slight acquaintance with combinatorial logic will easily understand why). Finally we define the relation of equivalence for derivations which convert HCC into a closed category, whose objects will be formulas, and whose morphisms will be equivalence classes of derivations. Instead of taking the relation $\equiv$ for equivalence classes $[d]$ of derivations $d$, we shall (as is often done in constructive mathematics) consider the derivations themselves, but understand the equality $=$ as $\equiv$. The reader who is not used to such «sets with equality» must change our $d = e$ to $[d] = [e]$, i.e. to $d \equiv e$.

We take as $\equiv$ the smallest equivalence relation which converts HCC into a closed category. More precisely $\equiv$ is defined inductively by the following axioms and rules 2.3. - 2.8.

2.3. General postulates

2.3.1. $a \equiv a$ ; $\quad \dfrac{a \equiv b \quad a \equiv c}{b \equiv c}$

2.3.2. $$\frac{a \equiv a' \quad b \equiv b'}{(a \supset b) \equiv (a' \equiv b')} \qquad \frac{a \equiv a' \; b \equiv b'}{ab \equiv a'b'} \qquad \frac{a \equiv a'}{\Pi^{-1}(a) \equiv \Pi^{-1}(a')}$$

2. 4. Postulates of category theory

$$(ab) \equiv a\,(bc); \qquad 1a \equiv a\,; \quad a\,1 \equiv a\,,$$

for $a, b, c$ with suitable types (that is, with suitable domains and codomains).

2.5. «Hom is a functor»

$(a_2a_1 \supset b_1b_2) \equiv (a_1 \supset b_1)\,(a_2 \supset b_2)$ for $A_2 \xrightarrow{a_1} A_1 \xrightarrow{a_2} A$, $B \xrightarrow{b_1} B_1 \xrightarrow{b_2} B_2$.

2.5.1. $1_{A \supset B} \equiv (1_A \supset 1_B)$.

2.6. Postulates of a closed category.

2.6.1. Invertibility of $i$. $ii^{-1} \equiv. 1\; i^{-1}\; i \equiv 1$.

2.6.2. Naturality of $i$. $(1 \supset b)i_A \equiv i_Bb$ for all $b : A \to B$.

2.6.3. Naturality of $j$. $(nj)$, chaging $=$ to $\equiv$.

2.6.4. Naturality of $L$L, see $(nL1)$, $(nL2)$, $(nL3)$.

2.7. Invertibility of $\Pi$. $\Pi(\Pi^{-1}(f)) \equiv f$; $\Pi^{-1}(\Pi(g)) \equiv g$ for $f : I \to A \supset B$, $g : A \to B$.

2.8. Conditions CC1-CC4, (4), (7).

We have not included the requirement that $\Pi^{-1}$ be natural, since a transformation which is the inverse of a natural one is itself natural. It is almost obvious that a closed category is actually obtained, in the sense of the second definition.

*Canonical morphisms* in a closed category are morphisms which are explicitly mentioned in the definition, that is, those which can be obtained from 1, $i$, $i^{-1}$, $j$, $L$ by hom, $\Pi^{-1}$ and composition.

We shall say that a sequent $A \to B$ is *valid* in closed categories (in brief, CC-valid), if for any closed category $K$ and substitution $\xi$ of objects from Ob $K$ for the propositional variables in $A$ and $B$, we have in Mor $K$ the canonical morphisms, $b : \xi\, A \to \xi\, b$.

Two canonical morphisms (regarded as formal combinations of the symbols $I$, $i$ etc.) are considered equal if their realizations in any closed category are equal. The main aim will be to give a decision procedure for CC-validity (to solve the word problem) and for the equality of canonical morphisms (to solve the coherence problem). The following lemma shows that it is sufficient to solve these problems for HCC.

LEMMA 2.1. $A \to B$ is deducible in HCC if and only if it is CC-valid. Two canonical morphisms $a$ and $b$ are equal if and only if $a \equiv b$.

*proof.* If $A \xrightarrow{b} B$ in HCC, then for any closed category $K$ and any substitution $\xi$ we have $b_\xi : \xi A \to \xi B$, where $b_\xi$ denotes the result of applying $\xi$ to the subscripts of the morphisms in $b$. Conversely, CC-validity of the sequence $A \to B$ denotes in particular that $b : A \to B$ holds in HCC. The second part of the lemma is proved analogously.

We note that by virtue of the lemma, it is impossible to have a situation where $A \to B$ is valid, but in different $K$ it is realized by different canonical morphisms: it is always possible to find a single morphism (one existing in HCC), which serves all $K$.

## 3. A System of Deductive Terms

We will describe a system of terms which will be used to study HCC. We assign to each term as its type some sequent of the form $\Gamma \to A$, where $\Gamma$ is a list (possibly empty) of formulas, and $A$ is a formula. The notation $t : \Gamma \to A$ or $\Gamma \xrightarrow{t} A$ means that $t$ is a term of the type $\Gamma \to A$. The terms will be constructed from constants I of type $\to I$ and of variables of type $A \to A$, for each formula $A$ denoted by $x^A, y^A, \ldots$ . The term $t : A_1, \ldots, A_n \to A$ will contain free exactly the (mutually distinct) variables $x^{A_1}, \ldots, x^{A_n}$.

DEFINITION. 1. $x^A : A \to A$ I: $\to I$.

2. If $a : \Pi \to A$, $b : \Gamma \to A \supset B$, then $(b,a)$: $\Gamma\ \Pi \to B$ on condition that $b$, $a$ have no common free variables.

3. If $b : \Gamma\ A \to B$ then $\lambda x^A\ b : \Gamma \to A \supset B$ where $x^A$ is a variable corresponding to an explicitly indicated occurrence of formula $A$ in $\Gamma A \to B$.

4. If $b : \Gamma\Sigma \to B$, $u : \Pi \to I$, then $\{b,u\} : \Gamma\Pi\Sigma \to B$ on condition that $b$, $u$ have no common free variables.

The first clause of the definition may be construed [18] as a representation of the axioms of the natural deduction calculus [10]. The remaining three clauses give, respectively, the rules for implication-elimination, implication-introduction, and (a limited form of) thinning. Thus the terms correspond to deductions in this calculus. Here the frequently used deduction

$$\frac{\Pi \xrightarrow{a} A\ ;\quad \Gamma\ A\ \Sigma \xrightarrow{e} E}{\Gamma\Pi\Sigma \to E}\ (\text{cut})\quad (e_{x^A}\ [a])$$

is modelled by the sequence of steps

$$\cfrac{\cfrac{\cfrac{\Gamma\ A\ \Sigma \xrightarrow{e} E}{\Gamma \xrightarrow{\lambda x^{A\Sigma} e} A \supset (\Sigma \supset E)} \qquad \Pi \xrightarrow{a} A}{\Gamma\Pi \xrightarrow{\lambda x^{A\Sigma} e,\ a} \Sigma \supset E, \qquad \Sigma \xrightarrow{x^{\Sigma}} \Sigma}}{\Gamma\ \Pi\ \Sigma \xrightarrow{((\lambda x^{A\Sigma} e,\ a),\ x^{\Sigma}} E}$$

where the double line indicates a series of implication-eliminations which contains the same number of terms as the list $\Sigma$. Under the familiar rules for operating with $\lambda$-symbols which we use below, a cut will correspond to the substitution indicated beside its conclusion.

The notation $a \doteqdot b$ indicates coincidence (graphical equality) of the syntactic expressions $a$, $b$; $t_x\ [a]$ is the result of substituting $a$ for all free (not bound by the symbol $\lambda$) occurrences of $x$ in $t$, the bound variables of $t$ being renamed to avoid collision. We shall not show $x$ when it is not needed.

We say that two terms are *congruent* if they are obtained from each other by the renaming of bound variables.

We define inductively an equivalence relation for terms.

$c \equiv \lambda\ x^A\ (c,\ x^A)$, if $c : \Gamma \to A \supset B$ and $x^A$ does not occur free in $b$; (1)

$t \equiv t'$ if $t$ is congruent to $t'$; (2)

$(\lambda\ x^A,\ b) \equiv a_{x^A}\ [b]$; (3)

$b_{x^I}\ [u] \equiv \{b_{x^I}\ [\mathrm{I}],\ u\}$, where $u : \mathrm{I} \to I$ and $x^I$ occurs free in $b$ ; (4)

$\{t,\ \mathrm{I}\} \equiv t$ ; $\{\mathrm{I},\ t\} \equiv t$ ; (5)

$$\frac{t \equiv t' \quad t \equiv t''}{t' \equiv t''} \qquad \frac{c \equiv c'}{t\ [c] \equiv t\ [c']} \tag{6}$$

We show below that $\equiv$ allows us to convert the set of formulas into a closed category, whose morphisms are equivalence classes of terms (of type $A \to B$) with respect to $\equiv$.

First, however, we introduce the translation, promised in the introduction, from the language of category theory to the language of proof theory.

| CATEGORY THEORY | PROOF THEORY |
|---|---|
| Category | Deductive system |
| Object | Formula |
| Morphism $f : A \to B$ | Derivation $f : A \to B$ |
| $1_A$ | Axiom $A \to A$ |
| Functor $F(A, B)$ | Logical connective $F(A, B)$ |
| covariant in $B$ | monotone in $B$ |
| contravariant in $A$ | antimonotone in $A$ with operation $F(a,b)$ on derivations $a : A_1 \to A$, $b : B \to B_1$, such that $F(a,b) : F(A,B) \to F(A_1,B_1)$ |
| Composition of morphisms $A \xrightarrow{b} B \xrightarrow{c} C$ | Cut $\dfrac{A \xrightarrow{b} B \quad B \xrightarrow{c} C}{A \to C}$ |
| $F(1_A, 1_B) = 1_{F(A,B)}$ | Axiom $F(A,B) \to F(A,B)$ can be broken into the axioms $A \to A$, $B \to B$. |
| $F(a_2\ a_1, b_1\ b_2) = F(a_1, b_1)\ F(a_2, b_2)$ | Cut on the formula $F(A_1, B_1)$ can be broken into cuts with respect to $A_1$ and $B_1$ |
| $\varphi_{AB} : F(A,B) \to G(A,B)$ is a natural transformation on $A$ of the functor $F$ to $G$ ($F$, $G$ are covariant) | $\varphi_A : F(A,B) \to G(A,B)$ is a derivation which can be permuted with «$G$-cuts» having cut-formula $A$ |

$$G\ (a,\ 1)\dfrac{\varphi_{AB}\ \dfrac{F\ (A,\ B)}{G\ (A,\ B)} \quad A \xrightarrow{a} A_1}{G\ (A_1,\ B)} \equiv \varphi_{A_1,B}\ \dfrac{\dfrac{F\ (A,\ B)\ A^a \to A_1}{F\ (A_1,\ B)}\ F\ (a,\ 1)}{G\ (A_1,\ B)}$$

We note that the translation of the concept «natural transformation» is less clear. This is perhaps to be expected, since in category theory this concept is generalized in a somewhat unexpected way to non-trivial situations, and different types of categories with additional structure (closed, monoidal, etc.) often differ from one another just by the stock of natural transformations.

## 4. The translation of combinators into terms and vice versa

We now assign to each canonical morphism $f : A \to B$ the corresponding term $\tau(f)$: $A \to B$

$$\tau\,(1_A) \doteq x^A; \qquad \tau\,(i_A) \doteq \lambda x^I\,\{x^A,\, x^I\} \qquad \tau\,(j_A) \doteq \lambda x^A\{x^A,\, x^I\}$$

$$\tau\,(i_A^{-1}) \doteq (x^{I\supset A},\, \mathrm{I}); \qquad \tau\,(L^A_{BC}) \doteq \lambda x^{A\supset B}\,\lambda x^A\,(x^{B\supset C},\, (x^{A\supset B},\, x^A));$$

$$\tau\,(a,\, b)) \doteq \tau\,(a)_{x^B}\,[\tau\,(b)] \qquad \text{for} \quad C \xrightarrow{b} B \xrightarrow{a} A$$

$$\tau\,((a \supset b)) \doteq \lambda\, x^{A_1}\,\tau\,(b)_{x^B}\,[(x^{A\supset B},\, \tau\,(a))] \quad \text{for} \quad A_1 \xrightarrow{a} A\ ,\quad B \xrightarrow{b} B_1$$

$$\tau\,(\Pi^{-1}\,(b)) \doteq (\tau\,(b)_{x^I}\ [I],\, x^A) \qquad \text{for} \quad b : I \to A \supset B$$

Thus, $\tau(a)$ is defined up to the renaming of free and bound variables.

To define the translation of terms into combinators we assign to each sequent $\Gamma \to A$ a corresponding formula $\Gamma \supset A$, and for each term $t : \Gamma \to A$ we define (by induction on the construction of $t$) a combinator $\boldsymbol{C}(t)$: $\mathrm{I} \to \Gamma \supset A$.

1. $\boldsymbol{C}(\mathrm{I}) \doteq 1_I\ ; \qquad \boldsymbol{C}(x^A) \doteq j_A\ ;$

2. $\boldsymbol{C}((b,\, a)) \doteq (1_\Gamma \supset i^{-1}_{\Pi\supset B}\ (\boldsymbol{C}\ (a) \supset 1_{\Pi\supset B}\ )\ L^\Pi)\ \boldsymbol{C}\ (b)$
   for $a : \Pi \to A,\ \ b : \Gamma \to A \supset B$ .

Explanation:

$$I \xrightarrow{C\,(b)} \Gamma A \supset B \xrightarrow{1_\Gamma \supset L^\Pi} \Gamma\,(\Pi \supset A)\,\Pi \supset B \xrightarrow{1_\Gamma \supset (C(a) \supset 1)} \Gamma\, I\, \Pi \supset B \xrightarrow{1_\Gamma \supset i^{-1}} \Gamma\Pi \supset B$$

When $\Gamma$ is empty, $1_\Gamma \supset$ and the corresponding brackets are missing, and for empty $\Pi$ we delete $L^\Pi$.

3. $\boldsymbol{C}\,(\lambda x^A\ b)) \doteq \boldsymbol{C}\,(b)$.

4. $\boldsymbol{C}\,(\{b,\, u\}) \doteq (1_\Gamma \supset i^{-1}\,(\boldsymbol{C}\,(u) \supset 1)\ L^\Pi)\ (1_\Gamma \supset \mathrm{I}_{\Sigma\supset B})\ \boldsymbol{C}\,(b)$
   for $b : \Gamma\Sigma \to B;\ \ u : \Pi \to I$.

Explanation. The definition is adapted to the equivalence

$$\frac{\Gamma\Sigma \xrightarrow{b} B \qquad \Pi \xrightarrow{u} I}{\Gamma\ \Pi\ \Sigma \to B} \equiv \frac{\Pi \xrightarrow{u} I \qquad \dfrac{\Gamma\Sigma \xrightarrow{b} B \quad I \to I}{\Gamma\ I\ \Sigma \to B}}{\Gamma\ \Pi\ \Sigma \to B}$$

which expresses thinning through its special case (where $\Pi = I$) together with a cut, more precisely

$$\mathbf{C}(\{b,u\}) \equiv \mathbf{C}((\lambda x^{I\Sigma}\ \{b,x^I\},ux^{\Sigma})) \tag{2}$$

where $\mathbf{C}(\{b,x^I\}) \equiv (1_\Gamma \supset i_{\Sigma\supset B})\mathbf{C}(b)$.

LEMMA 4.1. The sequent $A \to B$ is deducible in HCC if and only if there exists a term $t : A \to B$.

*Proof.* If $A \xrightarrow{a} B$ in an HCC, then $\tau(a)$: $A \to B$. If $t$ is a term of type $A \to B$, then $\Pi^{-1}$ $(\mathbf{C}(t))$ : $A \to B$ in HCC.

Our next problem is to show that the translation $\tau$ and $\Pi^{-1}\mathbf{C}$ are mutually inverse. Half of this assertion (lemma 4.2) is proved fairly easily, though clumsily, while the other half requires proof theory methods.

LEMMA 4.2. For any combinator $a : A \to b$

$$\mathbf{C}\ (\tau(a)) \equiv \Pi\ (a)$$

*Proof.* This is by induction on the construction of $a$, using the notation.

$$\tilde{a} \leftrightharpoons \mathbf{C}\ (\tau\ (a))$$

1. $\tilde{1}_A \equiv j_A$ ; $\Pi(1_A) \mathrel{\dot{=}} (1_A \supset 1_A)j_A \equiv 1_{A\supset A}j_A \equiv j_A$ .

2. $\tilde{i}_A \equiv (1_A \supset i_A)j_A \equiv \Pi\,(i_A)$.

3. $\tilde{j}_A \equiv j_{A\supset A}j_A \equiv \Pi(j_A)$ by CC5.

4. $\tilde{i}_A^{-1} \equiv i^{-1}_{(I\supset A)\supset A}\ ((1_{I\ \supset A} \supset i_A^{-1})j_{I\supset A} \supset 1)_{(I\supset A)\supset A}\ Lj_A \underset{9,\S 1}{\equiv} i^{-1}\ i\ \Pi\,(i_A^{-1})$ since $\Pi\ (i_A^{-1}) \equiv (1 \supset i_A^{-1})j_{I\ \supset A}$ .

5. $\tilde{L}^A_{BC} \equiv (1_{B\ \supset C} \supset i^{-1}_{(A\supset B)\ \supset C}\ \ (j_{A\supset B} \supset 1_{(A\supset B)A\supset C})\ L^{(A\supset B)A})j_{B\supset C}$

   $\underset{9,\S 1}{\equiv} (1 \supset i^{-1},\ i\ L^A)_j \equiv (1 \supset L^A)j \mathrel{\dot{=}} \Pi\ (L^A)$

6. $(\tilde{b}a) \equiv i^{-1}\ (\tilde{a} \supset 1)\ L^A\ \tilde{b} \equiv i^{-1}\ (\tilde{a} \supset 1)\ \Pi\ (1_A \supset b) \equiv$
   $\equiv i^{-1}\ (\tilde{a} \supset 1)\ (1 \supset (1 \supset b))j \equiv (1 \supset b)i^{-1}\ (\tilde{a} \supset 1)j \equiv$
   $\underset{9,\S 1}{\equiv} (1 \supset b)\ i^{-1}\ i\tilde{a} \equiv (1 \supset b)\ \tilde{a}$ .
   $\Pi\ (ba) \mathrel{\dot{=}} (1 \supset ba))j \equiv (1 \supset b)\ (1 \supset a)\ j \mathrel{\dot{=}} (1 \supset b)\ \Pi\ (a) \equiv$
   $(1 \supset b)\ \tilde{a}$ .

7. $\Pi^{-\tilde{1}}(b) \equiv i^{-1}(i^{-1}\tilde{b} \supset 1)j_{A\supset B} \underset{9,\S 1}{\equiv} i^{-1}\, i\, i^{-1}\tilde{b} \equiv$
$\equiv i^{-1}\tilde{b} \underset{9,\S 1}{\equiv} i^{-1}\, i\, b \equiv b \equiv \Pi(\Pi^{-1}(b))$

8. $a \tilde{\supset} b \equiv i^{-1}(a_1 \supset 1)L^{(A\supset B)A_1}\tilde{b}$, where $a_1 \equiv (1 \supset i^{-1}(\tilde{a} \supset 1)\, L^{A}_{1})j_{A\supset B}$.

Using naturality of L and CC2 we have:

$(\tilde{a} \supset 1)\, L \equiv (j \supset 1)\,((1 \supset a) \supset 1)\, L = (j \supset 1)\, L\, (a \supset 1) = i\, (a \supset 1);$

$a_1 \equiv (1 \supset (a \supset 1))j_{A\supset B} \equiv \Pi(a \supset 1)$

$a \tilde{\supset} b = i^{-1}(\Pi\,(a \supset 1) \supset 1)\, L^{A\supset B}\, L^{A_1}\, \Pi(b) \equiv$

$\underset{3,\S 1}{\equiv} i^{-1}(\Pi(a \supset 1) \supset 1)\, L^{A\supset B}\, \Pi(1 \supset b) \equiv$

$\equiv i^{-1}(j \supset 1)\,((1 \supset (a \supset 1)) \supset 1)\, L\Pi(1 \supset b) \equiv$

$\equiv i^{-1}(j \supset 1)\, L\,((a \supset 1) \supset (1 \supset b))j \equiv$

$\underset{6,\S 1}{\equiv} \Pi((a \supset 1)\,(1 \supset b)) \equiv \Pi(a \supset b)$

LEMMA 4.3. For any combinators $a, b : A \to B$ $a \equiv b$ implies $\tau(a) \equiv \tau(b)$, if identical free variables are chosen for the constructions $\tau(a)$ and $\tau(b)$.

*proof.* This is by induction on the derivation of the relation $a \equiv b$. This involves proving that if $a \equiv b$ is an axiom, then $\tau(a) = \tau(b)$ and if $a \equiv b$ is obtained from $a_1 \equiv b_1$, $a_2 \equiv b_2$, by the deduction rule, then $\tau(a) \equiv \tau(b)$ can be obtained from $\tau(a_i) \equiv \tau(b_i)$. We shall not carry out the proof, since it reduces to the writing out of long terms and calculations.

## 5. Conversion, reductions, and normal forms

Let $R$ be a binary relation between elements of some set. The *R-reduction* is the sequence (finite or infinite)

$$t_0, t_1, \ldots \tag{1}$$

where $t_i R t_{i+1}$ for all $i$. The sequence (1) is the $R$-reduction of $t$ to $t'$, if it is finite, $t_0 = t$, and the last term is $t'$. The notation $t \underset{R}{\models} t'$ is read «$t$ $R$-reduces to $t'$» and means that there is an $R$-reduction of $t$ to $t'$. The notation $t \underset{R}{\models} t'$ denotes $tRt'$. An element is *normal* (or, equivalently, is in $R$-normal form), if it will not reduce to any element except itself. We shall consider $R$ to be

fixed and omit reference to it. The simple theorem given below (compare [21]) is often used to establish that a normal form is unique up to inessential details.

THEOREM 5.1. Let (1) $E$ be some equivalence relation on the set which is being considered, and moreover suppose $t_1$, is normal if $t_0 E t_1$, and $t_0$ is normal;

(2) $R$ is well-founded, that is, all $R$-reductions are finite;

(3) if $t_0 E t_1$, $t_0 \vdash s_0$ $t_1 \vdash s_1$, then

$$\exists\, s'_0, s'_1\ (s'_0 \text{ E } s'_1 \quad \text{and} \quad s_0 \vdash s'_0 \quad \text{and} \quad s_1 \vdash s'_1)$$

Then the normal form always exists and is unique up to $E$, i.e.

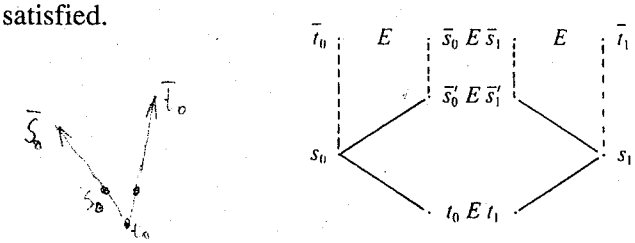
(4) If $t_0 E t_1$, $t_i \vdash \bar{t}_i$, and $\bar{t}_i$ are normal ($i = 0, 1$) it follows that $\bar{t}_0 E\ \bar{t}_1$.

*Proof.* By virtue of condition (2) of the theorem we can use induction on the tree of reduction of $t_0$ and $t_1$. More precisely $t_0$ and $t_1$ are two lowest nodes and if $s$ is located at node $a$, then the immediately preceding nodes contain exactly all the $s'$ such that $s \vdash s'$ (if such exist).

By virtue of condition (1) of the theorem we have (4) if either $t_0$ or $t_1$ is normal. This gives the basis for the induction, that is (4) for final vertices, since they contain normal terms. We justify the inductive step.

Let $t_i \vdash s_i$ **M** $\bar{t}_i$, ($i = 0, 1$). By virtue of (3) we find $s'_0$, $s'_1$ and by applying the inductive hypothesis to them, we get $\bar{s}_0\ E\ \bar{s}_1$ for the normal forms $\bar{s}_i$ of the terms $s'_i$, ($i = 0, 1$). We apply the inductive hypothesis to the pairs ($s_i$, $s_i$) and get $\bar{t}_i E \bar{s}_i$, ($i = 0, 1$). Since $E$ is transitive we get ($\bar{t}_0 E\ \bar{t}_i$), which was required.

We need a special case of this theorem, where $T$ is some set of terms or deductions and the relation $R$ is defined by means of some relation between the terms. Namely, we put $t \vdash t'$, if $t'$ is obtained from $t$ by changing its part $c$ to $c'$, where $c \vdash c'$ and some additional restrictions on $t$, $t'$, $c$,$c'$ are satisfied.




In such cases the relations $c \mapsto c'$ (and sometimes $t \vdash t'$, also) are known as *conversions*.

## 6. Normalization of terms. The decidability of the relation ≡

We consider the conversion

$$c \vdash \lambda\, x^A\, (c,\, x^A), \qquad \text{if } c : \Gamma \to A \supset B,\ x^A \text{ does not occur in c,} \quad (1)$$

$$c \neq \lambda y c' \qquad c \neq \{c',\, u\}$$

$R_1$ denotes the relation defined by the conversions (1) and the supplementary condition: the occurrence of the subterm $c$, which is changed to $\lambda x(c,x)$, does not originate from an occurrence of a subterm of the form

$$(\{c,u_1,\, u_2,\, \ldots,\, u_n\},\, d),\quad n \geq 0 \qquad (2)$$

where $\{c,\, u_1,\, \ldots,\, u_n\}$ denotes $\{\ldots\ \{\{c,u_1\},\, u_2\},\, \ldots,\, u_n\}$.

More explicitly, $t'\ R_1\ t''$ denotes that $t' = t_y[c]$, $t'' = t_y[\lambda x(c,x)]$ satisfy the conditions of (1) and the supplementary condition: the occurrence of $y$ in $t$ does not originate from an occurrence of a subtern of the form $(\{y, u_1,\, u_2, \ldots,\, u_n\},\, d)$. Occurrences of $c$ which satisfy (1) and this supplementary condition are *reducible*. $R_1$-normal terms are *reduced*. Let $E_1$ be the relation of congruence: $tE_1s$ denotes that $s$ is obtained from $t$ by renaming bound variables.

We will verify that the conditions of Theorem 5.1. are satisfied. (1) is obvious. We define the rank $\rho(A)$ of the formula $A$ as the number of occurrences of an implication in A. *The rank,* $\rho(t)$, of the term $t$, is by definition $\sum_j \rho(A_j)$ for all formulas of $A$ which correspond to reducible occurrences of the terms $c$ of (1), moreover for each such occurrence its own summand is added. It now follows that $R_1$ is well founded, since reduction lowers the rank, i.e. $tE_1\, t'$ implies $\rho(t') < \rho(t)$.

Condition (3) of Theorem 5.1. is verified by considering all possible cases of relative position of convertible subterms and indicating $s'_0$, $s'_1$ in each of these cases. So the following statement is proved.

THEOREM 6.1. Any term has a reduced form which is unique up to congruence.

The following series of conversions:

$$(\lambda xa,b) \mapsto a_x[b] \qquad (3)$$

$$c_x[v] \mapsto \{c_x[I],\, v] \text{ if } v \neq I \text{ and } x \neq x^I \text{ is in the scope of } \lambda \text{ or } (,) \qquad (4)$$

$$\{c,\mathrm{I}\} \mapsto c;\ \{\mathrm{I},c\} \mapsto c \qquad (5)$$

We denote by $R_2$ the relation defined by the conversions (3), (4), and (5) and the supplementary condition: the convertible terms are reduced. More precisely, $t' \; R \; t''$ denotes that $t' \doteqdot t_y[c']$, $t'' \doteqdot t_y[c'']$, $c' \mapsto c''$ and $t'$ is a reduced term.

We denote by $E_2$ the equivalence relation defined by renamig of bound variables, associativity, and commutativity of $\{\}$. More precisely, $E_2$ is defined inductively by the relation (6) - (10).

$$t \, E_2 \, \hat{t}, \quad \text{if} \quad t, \quad \hat{t} \quad \text{are congruent} \tag{6}$$

$$\{\{u, v\} \; w\} \; E_2 \; \{u \; \{v, w\}\} \; ; \quad \{u, v\} \, E_2 \; \{v, u\} \tag{7}$$

$$\frac{a \, E_2 \, b \qquad a \, E_2 \, c}{b \, E_2 \, c} \tag{8}$$

$$\frac{a \, E_2 \, b}{\lambda x a \, E_2 \, \lambda x b} \tag{9}$$

$$\frac{a E_2 \tilde{a} \;\; b E_2 \tilde{b}}{[a, \; b] \, E_2 \, [\tilde{a} \; \tilde{b}]} \tag{10}$$

where [] is either of the pairs () or $\{\}$.

LEMMA 6.2. If $aE_2\tilde{a}$ and $bE_2\tilde{b}$, *then* $a[b] \, E_2 \, \tilde{a} \, [\tilde{b}]$.

*Proof.* This begins by induction on the length of the term $a$ using (9) and (10) to prove that $b \, E_2 \, \tilde{b}$ implies $a[b] \; E_2 \; a[\tilde{b}]$.

Then we prove, by induction on the deduction of $a \, E_2 \, \tilde{a}$, that $a[\tilde{b}] \; \tilde{E}_2 \; \tilde{a}[\tilde{b}]$ and we apply the transitivity of $E_2$.

We proceed to prove the conditions of Theorem 5.1. Condition (1) is proved by induction on the demonstration of $t_0 \, E_2 \, t_1$.

In order to prove condition (2) we define a new rank $f_2(t)$ of the term $t$ as the sum $\rho_3(t) + \rho_4(t) + \rho_5(t)$. Here $\rho_3(t) = \omega \cdot \Sigma\rho(A \supset B)$ where the sum is taken over the formulas $A \supset B$ for the terms $\lambda x^A \, a : \Gamma \to A \supset B$ of all occurrences of subterms of type (3) up to $\{\}$, that is, subterms of the form

$$(\{\lambda x a, \; u_1, \; \ldots, \; u_n\}, \; b), \qquad n \geq 0 \tag{11}$$

$\rho_5(t)$ is the number of occurrences of type (5) terms.

For each occurrence $V$ of the term $v : \Pi \to I$ in $t$ we define as its *depth* the number of occurrences of $\lambda$ and ( ), in whose scope $V$ appears and from

which it can be «removed» according to conversion (4). Let $\rho_4(t)$ be the doubled sum of the depths of all such $V$. Now it is easy to prove that $t' R_2 t''$ implies $\rho_2(t'') < \rho_2(r')$. From this follows the second condition of Theorem 5.1. The third condition is proved as in the case of the reduction $R_1$, looking at all possible cases of relative positions of the convertible subterms. Together these give the uniqueness theorem.

THEOREM 6.3. Any term has a reduced $R_2$-normal form. This normal form is unique up to $E_2$.

This in turns gives the decision procedure for $\equiv$.

THEOREM 6.4. $t \equiv s$ is equivalent to $\bar{t}\ E_2\ \bar{s}$ for (any) normal forms $\bar{t}$, $\bar{s}$ of the terms $t$, $s$.

*Proof.* The implication right to left is obvious since the conversions and relations which define $E_2$, except commutativity and associativity of { }, are included in the postulates for $\equiv$. Commutativity is given by (4) and (5), and associativity follows from commutativity, (4) and (5).

We prove the implication left to right by showing that if $c \equiv c'$ is any of the relations (1) - (5) in § 3, then for any term $t'$ there are $\bar{t}, \bar{t}'$ such that $t[c] \models \bar{t}$ $t[c'] \models \bar{t}'$ and $\bar{t}\ E_2\ \bar{t}$.

We consider the corresponding cases. Here, because of symmetry we can assume that $c$ is in the form of the left hand side of the equivalence with the number we are investigating, and $\bar{c}'$ has the form of the right hand side. If for this particular occurrence of $c$ in $t$ the conditions for applying the corresponding conversions are satisfied, then we put $\bar{t} \doteq t[c'] \doteq \bar{t}'$. This settles cases (2), (3), and (5). We consider the remaining cases (1) and (4) of § 3.

(1) Let $c : \Gamma \to A \supset B$.

1.1. $c \doteq \lambda y b/$ We put $\bar{t} \doteq t[c]$; $\bar{t}' \doteq t[\lambda x b_y[x]]$.

1.2. $t[c] \doteq t[(c,d)]$. We put $\bar{t} \doteq \bar{t}' \doteq t$.

1.3. $c \doteq \{b, u_1, \ldots, u_n\}$, $n \geq 1$, moreover $b$ does not have the form $\{b', u\}$.

We make, if needed, the conversions (5) and we can assume that all $u_i$ are distinct from I. Further application of (4) and (5) reduces the problem to the cases 1.1. and 1.2.

(4) $u \doteq$ I or the occurrence of $u$ in $c[u]$ which we are considering is not in the scope of $\lambda$ or ( ).

If $u \doteq$ I, then $\{c\ [\mathrm{I}],\ u\} \mapsto c[\mathrm{I}] \doteq c[u]$.

If $u \neq$ I, then $\{c\ [\mathrm{I}],\ u\} \models_{(5)} \bar{c}$ where $\bar{c}\ E_2\ c\ [u]$. This is easily proved by induction on the depth of the occurrence of $u$ in $c$.

The theorem is proved.

## 7. The Correctness of Operation $C$

The main aim of this section is to prove the stability of the operation $C$ with respect to $\equiv$. The greatest difficulty is the relation $(\lambda xa,b) \equiv a_x[b]$. We will consider its generalization.

LEMMA 7.1. Let $e : \Gamma\ \Phi\ \Sigma \to E$, $f : X \to F$ be terms which do not have common free variables. Then

$$C\ ((\lambda x^F\ \lambda x^\Sigma\ e,f)) \equiv C\ (e_{x^F}[f]) \tag{1}$$

where $x^F$ is a variable of $e$ corresponding to the (explicitly indicated occurrence of) $F$, and $x^\Sigma$ is the list of variables corresponding to (the occurrences of the formulas of) $\Sigma$.

*Proof.* This is by induction on the construction of $e$.

1. $e \doteqdot x^F$. Then $C\ (\lambda x^F x^F) \doteqdot C\ (x^F) \doteqdot j_F$; $e[f] \doteqdot f$
$C\ (((\lambda x^F x^F,f)) \doteqdot i^{-1}\ (C\ (f) \supset 1)\ Lj_F \equiv i^{-1}\ (C\ (f) \supset 1) j_{x \supset F} \equiv$
$\underset{(9,\,\S 1)}{\equiv} i^{-1}\ i\ C\ (f) \equiv C\ (f)$

2. $e \doteqdot \lambda y^G h$. Then $C\ (e[f]) \doteqdot C\ (h[f]) \equiv C\ ((\lambda x^F\ \lambda x^\Sigma\ \lambda x^G\ h,f))$ by the inductive hypothesis, applied to $h$.

Before we consider further cases, we note that for any $e : \Gamma\ F \to E$

$$C((\lambda x^F\ e,\ x^F)) \equiv C(e) \tag{2}$$

Indeed, by CC2, $C((\lambda xe, x)) \doteqdot (1 \supset i^{-1}\ (j \supset 1)\ L)\ C(e) \equiv 1\ C(e) \equiv C(e)$.

3. $e \doteqdot (b,\ a)$.

3.1. $b : UFV \to A \supset B$, $a : Z \to E$. It is sufficient to prove that $f$ can be «advanced» to the subterm $b$ which contains $x^F$,

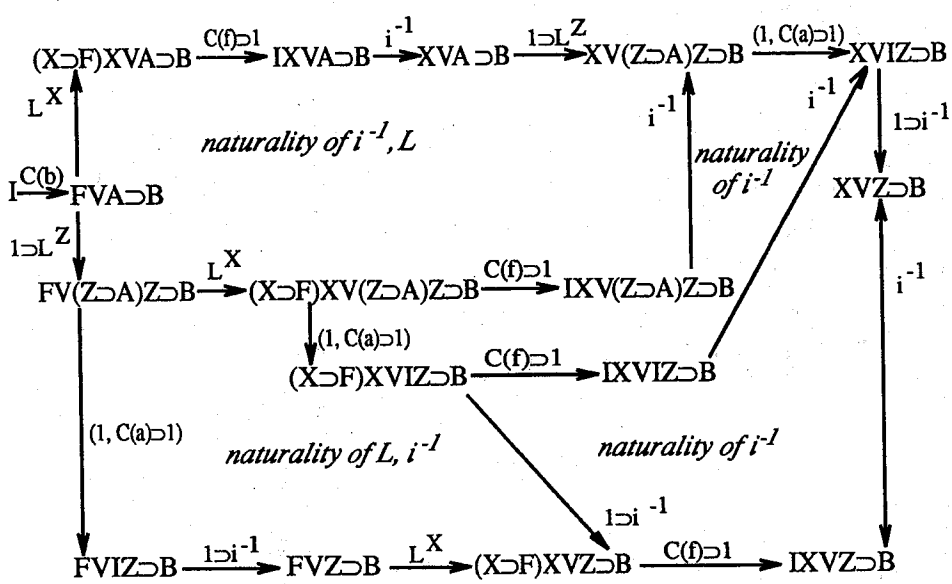
$$C((\lambda x^F\ \lambda x^{VZ}(b,\ a),\ f)) = C(((\lambda x^F\ \lambda x^V b,\ f),\ a)), \tag{3}$$

since the right hand side is transformed by the inductive hypothesis:

$$C((((\lambda x^F\ \lambda x^V b,\ f),\ a)) \doteqdot (1 \supset i^{-1}\ (C\ (a) \supset 1)\ L)\ C((\lambda x^F\ \lambda x^V b,\ f)) \equiv$$
$$\equiv (1 \supset i^{-1}\ (C(a) \supset 1)\ L)\ C(b[f]) \doteqdot C(e[f]).$$

In the commutative diagram given below it is assumed, for simplicity, that the list $U$ is empty (otherwise it would be necessary to add $1_U \supset$). The bor-

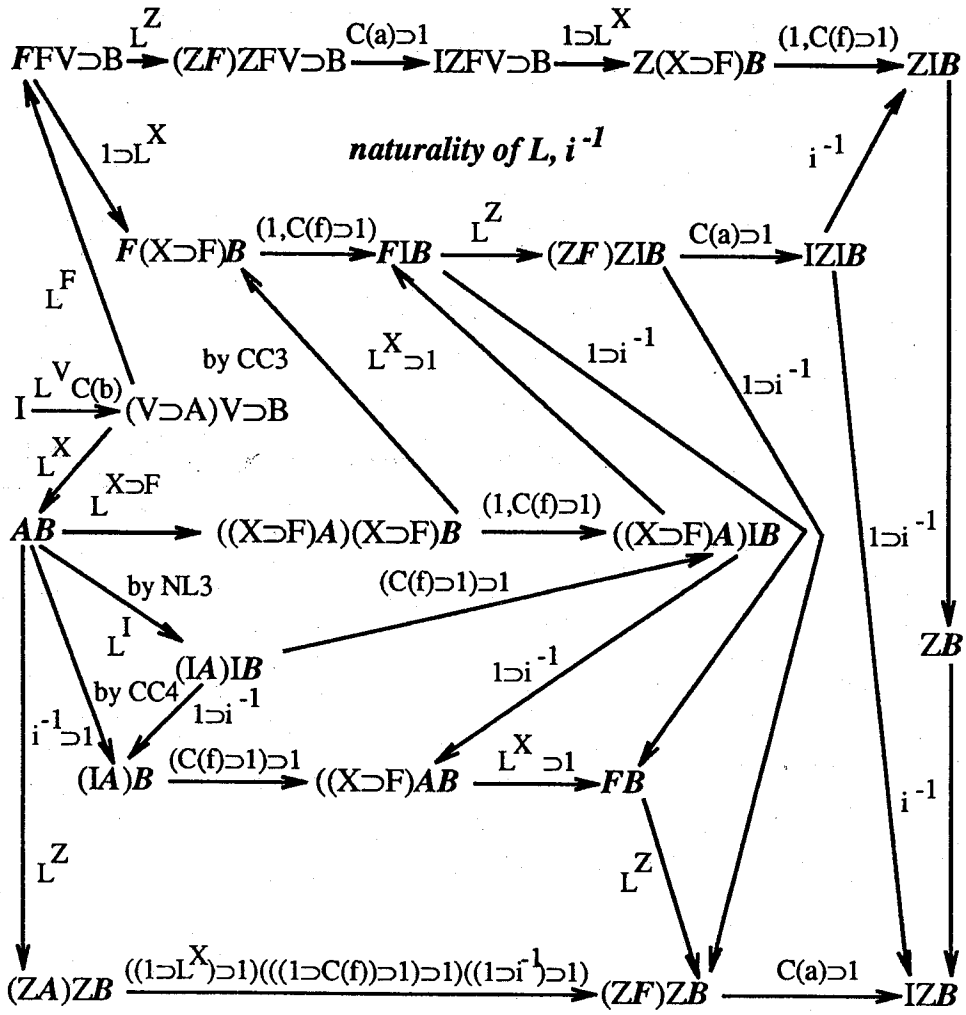
dering paths from $I$ to $XVZ \supset B$ are, respectively, $C(((\lambda x^F\ \lambda x^U b\, f), a))$ (the upper path) and $C((\lambda x^F\ \lambda x^{VZ} e, f))$ (the lower path). Inside each area of the diagram is indicated the principle by which that area commutes.




3.2. $b\colon U \to A \supset B$, $a : ZFV \to A$. As in 3.1 we prove.

$$C((\lambda x^F\ \lambda x^{VU}(b, a), f)) \equiv C((b, (\lambda x^F\ \lambda x^V a, f))) \tag{3}$$

with the help of the commutative diagram, where it is assumed that the list $U$ is empty and the notation $A \leftrightarrows XV \supset A$, $B \leftrightarrows XV \supset B$, $F \leftrightarrows FV \supset A$ is used




4. $e \doteq \{b,a\}$, where $b : \Gamma\Sigma \to B$ $a : \Pi \to I$.

We note that $\boldsymbol{C}(\{b,a\}) \equiv \boldsymbol{C}((\lambda x^I\ \{b,x^I\},a))$. If $x^F$ occurs in $a$ then we use the relation (4) and the inductive hypothesis. If $x^F$ occurs in $b$ then, after applying the relation (3), we can assume that $a \doteq x^I$, $\Pi \doteq I$. Now in the case $x^F \neq x^I$ it is sufficient to prove the possibility of transferring the permutations inside { }:

$$\boldsymbol{C}((\lambda x^F\ \lambda x^\Phi\ \{b,x^I\},f)) \equiv \boldsymbol{C}(\{(\lambda x^F\ \lambda x^{\Phi'}b,f),\ x^I\}), \tag{5}$$

if $x^F \neq x^I$ and $x^{\Phi'}$ is obtained from $x^\Phi$ by cancelling $x^I$ (if $x^F$ occurs in $x^\Phi$). In fact the lefthand side is equivalent to $(1 \supset i^{-1}\ (\boldsymbol{C}(f) \supset 1)\ L)\ (1 \supset i)\ \boldsymbol{C}\ (b)$, and the

right hand side is the composition $(1 \supset i)\ (1 \supset i^{-1}\ (\boldsymbol{C}(f) \supset 1)\ L)\ \boldsymbol{C}(b)$. We apply naturality.

In the case $x^I \doteqdot x^F$ we at once get $\boldsymbol{C}((\lambda x^I\ \lambda x^\Sigma\ \{b,\ x^I\},\ f)) \equiv \boldsymbol{C}(\{b,\ f\})$. The theorem is proved.

THEOREM 7.2. $d \equiv d'$ implies $\boldsymbol{C}(d) \equiv \boldsymbol{C}(d')$.

*Proof.* It is sufficient to consider the case where $d \equiv d'$ has one of the forms (1), (3)-(5), §3.

1. $\boldsymbol{C}\ ((\lambda x^A\ (b,\ x^A)) \doteqdot (1 \supset i^{-1}\ \ (j \supset 1)\ L)\ \boldsymbol{C}\ (b) \equiv \boldsymbol{C}\ (b)$.

3. $\boldsymbol{C}\ ((\lambda x^A(b,\ a)) \equiv \boldsymbol{C}\ (b[a])$ by lemma 7.1. with empty $\Sigma$.

4. $\boldsymbol{C}\ (b[u]) \equiv \boldsymbol{C}\ ((\lambda x^I\ \lambda x^\Sigma\ b,u))$;
$\boldsymbol{C}\ (\{b[\mathrm{I}],\ u\}) \equiv \boldsymbol{C}\ ((\lambda y^I\ \lambda x^\Sigma\ b,\mathrm{I}),y^I\},\ u))$.

therefore it is sufficient to prove

$$\boldsymbol{C}\ (\{(\lambda x^I\ \lambda x^\Sigma\ b,\mathrm{I}),y^I\}) \equiv \boldsymbol{C}\ (b). \tag{6}$$

5. $\boldsymbol{C}\ (\{b,\mathrm{I}\}) \equiv (1 \supset i^{-1})\ (1 \supset i)\ \boldsymbol{C}(b) \equiv \boldsymbol{C}\ (b)$
$\boldsymbol{C}\ (\{\mathrm{I},u\}) \equiv i^{-1}\ (\boldsymbol{C}\ (u) \supset 1)\ Li \equiv i^{-1}(1 \supset \boldsymbol{C}\ (u))\ i \equiv$
$\equiv \boldsymbol{C}\ (u)\ i^{-1}\ i \equiv \boldsymbol{C}\ (u)$.

The theorem is proved.

THEOREM 7.3. For any canonical morphisms $a$, $b$, $a \equiv b$ is equivalent to $\tau\ (a) \equiv \tau\ (b)$ which is equivalent to $\overline{\tau\ (a)\ E_2\ \tau\ (b)}$, where the bar indicates reduction to the normal form and $E_2$ is the equivalence relation from § 6.

*Proof.* By theorems 4.2, 7.2, and 6.4.

## 8. The Coherence Theorem

Let $A^* = (A \supset I)$. We consider two morphisms

$$\begin{tikzcd}
(P^* \supset P^*)^* \arrow[r, "jp^* \supset 1_I"] \arrow[dr, "\mathrm{I} \supset j_{P^*}"'] & I \supset I \arrow[r, "i^{-1}"] & I \arrow[d, "j_{(P^* \supset P^*)}"] \\
& & (P^* \supset P^*) \supset (P^* \supset P^*)
\end{tikzcd} \tag{1}$$

and corresponding terms

$$t_1 \leftrightharpoons \lambda x^2\ \lambda x^1\ \{x^1,\ (x^3,\ x^2)\}; \qquad t_2 \leftrightharpoons \lambda x^2\ \{x^2,\ (\lambda x^1\ (x^3,\ \lambda x^1\ \{x^1,\ x^{I})),\ I)\}$$

where we put $x^1 \leftrightharpoons x^{P*}$ ; $x^2 \leftrightharpoons x^{P*\supset P*}$ ; $x^3 \leftrightharpoons x^{(P*\supset P*)*}$.

We construct the normal forms

$$t_1 \equiv \lambda x^2\ \{\lambda x^1\ x^1,\ (x^3\ x^2)\} \equiv \lambda x^2\ \{\lambda x^1\ \lambda x^P\ (x^1, x^P),\ (x^3,\ \lambda x^1\ (x^2,\ \lambda x^P\ (x_1, x^P)))\}$$

$$t_2 \equiv \lambda x^2\ \{x^2,(x^3,\ \lambda x^1\ x^1)\} \equiv \{\lambda x^2\ x^2,(x^3,\ \lambda x^1\ x^1)\} \equiv$$
$$\equiv \lambda x^2\ \lambda x^1(x^2,\ \lambda x^P(x_1,\ x^P)),\ (x^3,\ \lambda x^1\lambda x^P\ (x^1,\ x^P))\}$$

We see that the normal form $t_1$ begins with $\lambda$, and the normal form $t_2$ is of type $\{u,v\}$ so that $t_1 \not\equiv t_2$.

Thus diagram (1) is non-commutative. However the morphisms which constitute it are special cases of distinct natural transformations:

$$(q \supset q)^* \xrightarrow{jq\supset 1} I \supset I \xrightarrow{i^{-1}} I \xrightarrow{j_r} r \supset r$$

$$s^* \xrightarrow{1\supset j_u} s \supset (u \supset u) \tag{2}$$

where $u \mathrel{\dot{=}} q = p^*$ $s \mathrel{\dot{=}} r \mathrel{\dot{=}} p^* \supset p^*$.

By modifying Lambek's method, we now show that if the variables are not identified, then all the diagrams will be commutative.

Let us say that an occurrence of a variable in a formula is positive (negative) if it occurs in an even (odd) number of premises of implications. The sign of occurrence in $\Gamma \to A$ is the same as in $\Gamma \supset A$. Let us suppose that the sequence is *balanced* if each variable occurring in it has exactly 2 occurrences and they have opposite signs. The term $t : \Gamma \to A$ is balanced if the sequence $\Gamma \to A$ is balanced. We want to prove that balanced terms of the same type are equivalent.

We prove first that balanced terms are in a certain sense the general case.

THEOREM 8.1. Every normal term $t$ is obtained by substitution of formulas for variables in some balanced term.

*Proof.* This is by induction on the length of $t$. The axioms and one-premise rules are considered in the obvious way. In the case of a two-premise rule we only need to choose the disjoint lists of variables for the balanced sequences which correspond to premises by inductive hypothesis. The theorem is proved.

LEMMA 8.2. If $t : \Omega\Pi\Sigma \to A$ and list $\Pi$ have no variables in common with $\Omega$, $\Sigma$, $A$ then $t \equiv \{a,u\}$, where $a : \Omega\Sigma \to A$, $u : \Pi \to I$.

*Proof.* If the list $\Pi$ is empty, we put $a \doteqdot t$, $u \doteqdot \mathrm{I}$. Let us assume that $\Pi$ is not empty, and apply induction on the length of the sequent $\Omega\Pi\Sigma \to A$. We suppose $t$ to be normal. It can be a variable of type $x^I$ only, then $t \equiv \{\mathrm{I}, x^I\}$. If $t$ is not a variable we apply the inductive hypothesis to one or two subterms and then use (4), §3. For example if $t = (t_1 t_2)$, $t_i = \{a_i, u_i\}$ then $t = \{a_i, a_2\}$, $\{u_1, u_2\}$. The theorem is proved.

THEOREM 8.3. If $t, t' : \Gamma \to A$ are balanced terms, then $t \equiv t'$.

*Proof.* Let $t$, $t'$ be normal, and apply induction on the length $l(t)$ of the term $t$. If $\Gamma \doteqdot \Omega\Pi\Sigma$, the sequent $\Pi \to I$ is balanced, and $\Pi$ is non-empty, we can apply lemma 8.2 and the inductive hypothesis. Therefore we may assume in future that, when $\Pi$ is non-empty, the hypotheses of lemma 8.2 are not satisfied so that in particular neither $t$ nor $t'$ is of the form $\{a,u\}$. If one of $t$, $t'$ begins with $\lambda$, then by virtue of normality, it is also true of the other, and again we can apply the inductive hypothesis. If either $t$ or $t'$ is not a variable, then, since they are normal, neither is the other. There remains the case

$$t \doteqdot (\ldots (a,b_1) \ldots, b_n),\ t' \doteqdot (\ldots (a', b'_1) \ldots, b'_m), \tag{3}$$

where

$$a \doteqdot x^{B_1 \ldots B_n \supset A}, \quad a' \doteqdot x^{B'_1 \ldots B'_m \supset A'}$$

$$b_i : \Gamma_i \to B_i; \quad b_j : \Gamma'_j \to B'_j \quad (i \leq n,\ j \leq m).$$

From (4) and the fact that the upper index of the variables $a$, $a'$ is the first member of the list $\Gamma$, it follows that $n = m$, $a \doteqdot a'$, $B_i \doteqdot B'_i$ $(i = 1, \ldots, n)$.

We prove that $\Gamma_n \doteqdot \Gamma'_n$, after which we can apply the inductive hypothesis. Indeed, if, for example, we had $\Gamma_n \doteqdot \Pi\Gamma'_n$ with non-empty $\Pi$ then the sequent $\Pi \to I$ would be balanced in spite of our assumption that lemma 8.2 is inapplicable. The theorem is proved.

Now from theorems 8.3 and 7.3 we obtain the coherence theorem.

THEOREM 8.4. If $a$, $a' : B \to A$ are canonical morphisms and the sequent $B \to A$ is balanced, then $a = a'$.

A comparison of the proof of theorem 8.3 with the proofs in [7] and [1] shows the advantage of the method of deductive terms in these problems.

## 9. Symmetric Closed Categories

Symmetry means the presence of a natural isomorphism

$$C'_{ABD} = C'_{AB} = C' : AB \supset D \to BA \supset D$$

which satisfies the conditions:

$$C'_{AB}\, C'_{BA} = 1 \tag{1}$$

$$L^y\, C'_{AB} = (1 \supset C'_{Ay})\, C'_{A(y\supset B)}\, (1 \supset L^y) : A\, B \supset D \to (y \supset b)\, yA \supset D \tag{2}$$

$$1_A \supset i_D^{-1} = i^{-1}_{A\supset E}\, C'_{AI} \tag{3}$$

We shall show how to supplement the proofs from the preceding paragraphs in order to obtain proofs of all the assertions, except for theorems 8.3 and 8.4. These theorems are invalid, as the example introduced in [1] shows

$$\begin{tikzcd} A^{***} \arrow[r, "K_A \supset 1"] \arrow[dr, "1"'] & A^{*} \arrow[d, "K_{A^*}"] \\ & A^{***} \end{tikzcd} \qquad K_A = \Pi^{-1} \cdot (C'_{A^*A}\, j_{A^*}) : A \to A^{**}$$

We also outline the proof of a weaker theorem of coherence, analogous to the theorem in [1].

Another clause is added to the definition of a term in §3: if $t : \Gamma AB\Sigma \to D$ then $t : \Gamma AB\Sigma \to D$. In the definition of the operation $\boldsymbol{C}$ we should now indicate the order of the formulas to the left of the arrow; in particular

$$\boldsymbol{C}_{\Gamma BA\Sigma}(t) \doteqdot (1_\Gamma \supset C'_{AB})\, \boldsymbol{C}_{\Gamma AB\Sigma}(t)$$

In the remaining clauses of the definition of $\boldsymbol{C}$ we simply add a subscript, in particular

$$\boldsymbol{C}_\Gamma(\lambda x^A s) \doteqdot \boldsymbol{C}_{\Gamma A}(s).$$

We put $\tau(C'_{ABD}) \doteqdot \lambda x^B\, \lambda x^A((x^{AB\supset D}, x^A), x^B)$.

Actually the only place where an addition is required is the proof of lemma 7.1, where the naturality of $C'$ and the relations (2) and (3) are used.

In order to formulate the analog to theorem 8.4, we call a formula *constant* if it does not contain variables (i.e. it is constructed from the constant $I$). We say that the formula or sequent is *I-balanced* if it is balan-

ced and in any of its subformulas $(A \supset C)$ with constant $C$ the antecedent $A$ is also constant. In [1] the analogous objects are called *proper shapes*.

THEOREM 9.1. If $t : \Omega \to A$, $t' : \Omega' \to A$, the lists $\Omega$, $\Omega'$ differ only in the order of their terms, and $\Omega \to A$ is I-balanced, then $t \equiv t'$.

*Proof.* This is analogous to the proof of theorem 8.3, but uses an additional lemma (compare assertion 7.4 in [1]) which is easily proved by induction using the properties of normal form.

LEMMA 9.2. If $t : \Gamma \to C$ where $\Gamma \to C$ is an I-balanced sequent and $C$ is a constant formula, then $\Gamma$ consists of constant formulas.

In examining the main case in the proof of theorem 9.1 (cf. (3), §8), instead of referring to the unique ordering of the formulas in the lists $\Omega$, $\Omega'$ (this may be invalid due to the formulation of theorem 9.1), we turn to lemma 9.2 to establish that $A$ is a variable (and not the constant $I$) and then appeal to balance, in order to prove that the upper indices in (4), §8 coincide.

## 10. Monoidal closed categories

We shall not give the definition from [2] in full. We note that in a monoidal category there is a two-place covariant functor $\otimes$ (the tensor product) and natural isomorphisms

$$r_A : A \otimes I \to A;\ l_A : I \otimes A \to A,\ a_{ABC} : (A \otimes B) \otimes C \to A \otimes (B \otimes C).$$

Moreover for any two morphisms $d, d' : A \to B$, constructed from $a$, $r$, $l$ by means of $\otimes$ and composition, $d \equiv d'$ holds, if $A \to B$ is a balanced sequent and $A$, $B$ are constructed from $I$ and variables using $\otimes$.

A *monoidal closed category* is a closed category which is at the same time monoidal, $\otimes$ and $\supset$ being connected by a natural isomorphism

$$P = P_{ABC} : A \otimes B \supset C \to A \supset (B \supset C) \tag{1}$$

which satisfies the relations:

$$i_{A \supset B} = P_{IAB}(l_A \supset 1_B) \tag{2}$$

$$1_A \supset i_B = P_{AIB}(r_A \supset 1_B) \tag{3}$$

$$(1 \supset P_{BCD})P_{A(B \otimes C)D} = P_{AB(C \supset D)}P_{(A \otimes B)CD}(a_{ABC} \supset 1) \tag{4}$$

$$L^B_{AC} = P_{(B \supset A)BC}\ (\Pi^{-1}\ (P^{-1}j \supset 1_c) \tag{5}$$

All the results of section 1-8 are also true for canonical morphisms in such categories, including the coherence theorem 8.4.

We will add two clauses to the definition of a term:

if $e : \Gamma AB\Sigma \to E$, $f : \Pi \to A \otimes B$, then $e_{x^A x^B}[\mathbf{l}f.\mathbf{r}f] : \Gamma\Pi\Sigma \to E$.

If $f : \Pi \to A$, $g : \Phi \to B$, then $(f,g) : \Pi\Phi \to A \otimes B$ with the condition that $e$, $f$, $g$ have no common free variables.

The definition of the operation $\tau$ is augmented by the following clauses:

$$\tau\,(r_A) \doteq \{l_x\ \boldsymbol{r}\ x\} : \tau\,(l_A) \doteq \{\boldsymbol{r}\ x,\ \boldsymbol{\ell}\ x\} \qquad \text{where} \quad x \doteq x^{\,A\otimes I}$$

$$\tau\,(a_{ABC}) \doteq \langle \boldsymbol{\ell\ell}\ x,\ \langle \boldsymbol{r}\ l\ x,\ \boldsymbol{r}\ x\ \rangle\rangle, \qquad \text{where} \quad x \doteq x^{\,(A\otimes B)\otimes C}$$

$$\tau\,(P_{ABC}) \doteq \lambda x^A\ \lambda x^B\ (x^{A\otimes B\supset C},\ \langle x^A,\ x^B\rangle).$$

The definition of operation $\boldsymbol{C}$ is supplemented by the clause:

$$\boldsymbol{C}\,(e\ [\boldsymbol{l}_f\,,\ \boldsymbol{r}\ f\ ]) \doteq 1_\Gamma \supset i^{-1}\ (\boldsymbol{C}\,(f) \supset 1)\ L^\Phi\ P^{-1})\ \boldsymbol{C}\,(e).$$
$$\text{for} \quad \Gamma AB\ \Sigma \xrightarrow{e} E,\ \Phi \xrightarrow{f} A \otimes B$$

$$\boldsymbol{C}\,(\langle f,\ g\ \rangle) \doteq \Pi\,((1_\Gamma \supset P^\Phi)\ P^\Gamma\ P\,(\Pi^{-1}))^\Gamma\ \boldsymbol{C}\,(f\ )) \otimes \Pi^{-1}((P^{-1})^\Phi\ \boldsymbol{C}\,(g)))).$$
$$\text{for} \quad f : \Delta \to A,\ g : \Phi \to B$$

The following clauses are added to the definition of equivalence:

$$f \equiv \langle \boldsymbol{l}\ f,\ \boldsymbol{r}\ f\ \rangle \qquad \text{for} \quad f : \Pi \to A \otimes B$$

$$e_{x^A x^A}\,[\boldsymbol{l}\,\langle f, g\rangle, \boldsymbol{r}\ \langle f, g\rangle] \equiv e_{x^A x^B}\,[f, g] \text{ for } f : \Pi \to A,\ g : \Phi \to B,\ e : \Gamma\ \ AB\Sigma \to E$$

The proof of the theorem on the existence and uniqueness of a normal form and the coherence theorem are obtained by extending the corresponding proofs for closed categories. There is an important additional argument: if $\Gamma \to A$ and $\Pi \to B$ have no common variables, $t : \Pi\Sigma \to A \otimes B$ is normal and does not have the form $\{a,u\}$, then $t = \langle a,b\rangle$ for $a : \Gamma \to A$, $b : \Sigma \to B$.

Finally, symmetric monoidal closed categories which were examined in [1] can be treated by means of all these techniques: the equality of canonical morphisms is reduced to equivalence (of normal forms) of terms, and the theorem of coherence for I-balanced morphisms and terms is proved as in §9.

We note that the theorem about the normal form for canonical morphisms $f : A \to B$ for $A$, $B$ which do not contain $I$, in symmetric monoidal closed categories, is equivalent to the one proved in [22] on cut elimination for a

formal system which is obtained by excluding the contraction rule from the $(0, \supset)$-fragment of the relevant logic $R$. Actually [22] deals with the calculus $R^+$, a positive fragment of the logic $R$, containing as connectives $\{0, \supset, \vee, \&)$, as well as the contraction rule. However the theorem on cut-elimination in [22] is formulated in a way that gives us the result we are interested in: reductions do not introduce applications of the contraction rule (cf. also [23]). Thus the problem posed in [13], of obtaining the theorems in [1] by direct application of Gentzen methods, was solved before it was set. The author learnt of the work [1] only from [13].

## CITED LITERATURE


[1] G. Kelly, S. MacLane, «Coherence in closed categories», in *J. of Pure and Appl. Algebra*, 1, No. 1, 97-140, 1971.

[2] S. Eilenberg, G. Kelly, «Closed categories», in *Proc. Conf. on Categorical Algebra La Jolla*, 1965, Springer-Verlag (1966), 421-562.

[3] G.E. Mints, «Proof theory and Category theory», This volume.

[4] G.E. Mints, «Normalization of proofs», Appendix to book by S.C. Kleene, *Mathematical Logic* (in Russian), M., 1973.

[5] D. Prawitz, *Natural Deduction, a Proof Theoretical Study*, Stockholm, 1965.

[6] G.E. Mints, «Proof Theory (Arithmetic and Analysis)» in *Algebra, Topology, Geometry*, Vol. 13, Viniti., Moscow (1975), 5-49. (English Translation: *Journal of Soviet Math.*, v. 7, N. 4, 501-531, 1977).

[7] J. Lambek, «Deductive systems and categories I», in *Math. Systems Theory*, *2*, No. 4, 287-318, 1968.

[8] J. Lambek, «Deductive systems and categories II», in *Lecture Notes in Mathematics*, *86*, 76-122, 1969.

[9] J. Lambek, «Deductive systems and categories III», in *Lecture Notes in Mathematics*, *274*, 57-82, 1972.

[10] D. Prawitz, «Ideas and Results in Proof Theory», in *Proc. 2nd Scand. Logic Sympos.*, Amsterdam, 1971, 235-307.

[11] G. Kreisel, «A survey of proof theory II», in *Proc. 2nd Scand. Logic Sympos.*, Amstedam, 1971, 109-170.

[12] K. Schütte, *Beweistheorie*, Berlin, 1960.

[13] S. MacLane, «Topology and Logic as a source of algebra», in *Bull. Amer. Math. Soc.*. *82*, No. 1, 1-40, 1976.

[14] J. Zucker, «The correspondence between cut-elimination and normalization», in *Ann. of Math. Log.*, *7*, No. 1, 1-112, 1974.

[15] M. Szabo, «A categorical equivalence of proofs», in *Notre Dame J. Form. Log.*, *15*, No. 2, 171-191, 1974.

[16] M. Szabo, Addendum (to [15]), *Notre Dame J. Form. Log.*

[17] C. Mann, «The connection between equivalence of proofs and cartesian closed categories», in *Proc. London Math. Soc.*, *31*, No. 3, 289-310, 1975.

[18] P. Martin-Löf, «Infinite terms and a system of natural deduction», in *Compos. Math.*, *24*, No. 1, 93-103, 1972.

[19] G. Kelly, «An abstract approach to coherence», in *Lecture Notes in Mathematics*, *281*, 106-147, 1972.

[20] S. Eilenberg, G. Kelly, «A generalization of the functorial calculus», in *J. Algebra*, *3*, No. 3, 366-375, 1966.

[21] A. Troelstra (ed.), «Metamathematical investigations of intuitionistic arithmetic and analysis», in *Lecture Notes in Mathematics*, *344*, 1974.

[22] G.E. Mints, «A theorem on cut-elimination for relevant logic» (Russ.), in *Zap. Nauchn. Seminarov Leningr. otd. Matem. in-ta AN SSSR*, *32*, 90-97, 1972.

[23] G.E. Mints, «The independence of postulates of natural calculus», in *Zap. Nauchn. Seminarov Leningr. otd. Matem. in-ta AN SSSR*, *8*, 912-915, 1968.