Survey of Voting Procedures and Paradoxes

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The Voting Problem

Given a (finite) set X of **candidates**

and a (finite) set A of **voters**

each of whom have a **preference** over X (for simplicity, assume a connected and transitive)

devise a method F which aggregates the individual preferences to produce a collective decision (typically a subset of X).

Voting Procedures

- *Roughly* three different types of procedures: ranked, non-ranked, multi-stage.

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- Each procedures specifies a type of vote, or **ballot**, that is recognized as admissible by the procedure and a method to **count** a vector of ballots (one ballot for each voter) and select a winner (or winners).

Many Examples

Plurality (Simple Majority)

- Each voter selects one candidate (or none if voters can abstain)
- The candidate(s) with the most votes wins.

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Negative Voting

- Every voter can select one candidate to voter for or against.
- The candidate(s) with the most votes wins.

(Equivalent to either giving one vote to a single candidate or one vote to everyone but one candidate)

Many Examples

Approval Voting

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Cumulative Voting

- Every voter is given k votes which can be cast arbitrarily (several votes for the same candidate are allowed)
- The candidate(s) with the most votes wins.

Many Examples

Plurality with runoff

- Use plurality voting to select the winner(s)
- If two or more candidate tie for the win, they move on to round two. If there is a unique winner in round 1, that candidate and the second place winner(s) move on to round two.
- Use plurality vote on this smaller set of candidates.

(More generally, alternative rules can be used to determine who moves on to the next round)

Many Examples

Pairwise Elimination

- In advance, voters are given a schedule for the order in which pairs of candidates will be compared.
- In the above order, successively eliminate the candidates preferred by a minority of votes.
- The winner is the candidate who survives.

Many Examples

Borda Count

- Each voter provides a linear ordering of the candidates.
- The candidate(s) with the most **points** wins, where points are calculated as follows: if there are n candidates, $n-1$ points are given to the highest ranked candidates, $n-2$ to the second highest, and so on.

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The Hare System

- Each voter provides a linear ordering of the candidates.
- Repeatedly delete the candidate or candidates with the least first-place votes. The last group to be deleted is tied for the win.

Comparing Voting Procedures

Arrow's Theorem shows use that with more than three choices, there is no "perfect" procedures. How should we compare the procedures?

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- Is the procedure **monotonic**? More votes should always be better!

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- How *expressive* are the ballots? How practical is the system to implement?
- A **Condorcet winner** is a candidate that beats every other candidate in pairwise contests. A voting procedure is Condorcet provided it selects the Condorcet winner, if one exists.
- Is the procedure **monotonic**? More votes should always be better!
- How susceptible is the procedure to *manipulation*?

Failure to elect the Condorcet candidate

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

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Condorcet Winner: c beats each candidate in a pairwise comparisons.

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Plurality: a is the plurality winner.

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# voters	3	5	7	6
3	a	a	b	c
2	b	c	d	b
1	c	b	c	d
0	d	d	a	a

Borda:

- $BC(a) = 3 \times 3 + 3 \times 5 + 0 \times 7 + 0 \times 6 = 24$
- $BC(b) = 2 \times 3 + 1 \times 5 + 3 \times 7 + 2 \times 6 = 44$
- $BC(c) = 1 \times 3 + 2 \times 5 + 1 \times 7 + 3 \times 6 = 29$
- $BC(d) = 0 \times 3 + 0 \times 5 + 2 \times 7 + 1 \times 6 = 20$

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Failure to elect the Condorcet candidate

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
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	d	d	a	a

Condorcet: *c* beats each candidate in a pairwise comparisons.
Plurality: *a* is the plurality winner.
Borda: *b* is the Borda winner.

Scoring Rules

Fix a nondecreasing sequence of real numbers

$$s_0 \leq s_1 \leq s_1 \leq \cdots \leq s_{m-1}$$

with $s_0 < s_{m-1}$

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Voters rank the candidates, giving s_0 points to the one ranked last, s_1 to the one ranked next to last, and so on. A candidate with the maximal total score is elected.

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Theorem (Fishburn) There are profiles where the Condorcet winner is never elected by **any** scoring method.

AV is more flexible

Fact There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	a	b	c
	d	d	a
	b	a	b
	c	c	d

AV is more flexible

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# voters	2	2	1
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The unique Condorcet winner is *a*.

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# voters	2	2	1
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Vote-for-1 elects $\{a, b\}$, vote-for-2 elects $\{d\}$, vote-for-3 elects $\{a, b\}$.

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# voters	2	2	1
	a	b	c
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$(\{a\}, \{b\}, \{c, a\})$ elects a under AV.

AV is more flexible

Fact Condorcet winners are always AV outcomes, but a Condorcet looser may or may not be an AV outcome.

The Spoiler Effect

# voters	35	33	32
	a	b	c
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# voters	35	33	32
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Plurality and Borda both pick *a*.

The Spoiler Effect

# voters	35	33	32
	a	b	c
	c	a	b
	b	c	a

Candidate *c* is a spoiler.

The Spoiler Effect

# voters	35	33	32
	a	b	x
	x	x	b
	b	c	a

Without c, both Plurality and Borda both pick b.

Failure of Monotonicity

# voters	6	5	4	2
	a	c	b	b
	b	a	c	a
	c	b	a	c

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The profiles are monotonic (in *a*).

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# voters	6	5	4	2
	a	x	b	b
	b	a	x	a
	x	b	a	x

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No-show Paradox

Totals	Rankings	H over W	W over H
417	B H W	417	0
82	B W H	0	82
143	H B W	143	0
357	H W B	357	0
285	W B H	0	285
324	W H B	0	324
1608		**917**	**691**

Fishburn and Brams. *Paradoxes of Preferential Voting*. Mathematics Magazine (1983).

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B: $417 + 82 = 499$
H: $143 + 357 = 500$
W: $285 + 324 = 609$

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H Wins

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Suppose two more people show up with the ranking B H W

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W Wins!

Multiple Districts

Totals	Rankings	East	West
417	B H W	160	257
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1608		**588**	**1020**

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B would win both districts!

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Young's Theorem

Reinforcement: If two disjoint groups of voters N_1 and N_2 face the same set of candidates and N_i selects B_i. If $B_1 \cap B_2 \neq \emptyset$, then $N_1 \cup N_2$ should select $B_1 \cap B_2$.

Continuity Suppose N_1 elects candidate a and a disjoint group N_2 elects $b \neq a$. Then there is a n such that $(nN_1) \cup N_2$ chooses a.

Theorem (Young) A voting correspondence is a scoring method iff it satisfies anonymity, neutrality, reinforcement and continuity.

Young. *Social Choice Scoring Functions*. SIAM Journal of Applied Mathematics (1975).

Approval Voting

Theorem (Fishburn) A voting correspondence is approval voting iff it satisfies anonymity, neutrality, reinforcement and

> *If a profile consists of exactly two ballots (sets of candidates) A and B with $A \cap B = \emptyset$, then the procedure selects $A \cup B$.*

Fishburn. *Axioms for Approval Voting: Direct Proof.* Journal of Economic Theory (1978).

The Danger of Manipulation

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Setting the Agenda:

# voters	35	33	32
	a	b	c
	c	a	b
	b	c	a

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The order: 1. a vs. b; 2. the winner vs. c elects c

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The order: 1. b vs. c; 2. the winner vs. a elects a

The Danger of Manipulation

Setting the Agenda:

# voters	1	1	1
	b	a	c
	d	b	a
	c	d	b
	a	c	d

The order: 1. a vs. b; 2. the winner vs. c; 3. the winner vs. d elects d

The Danger of Manipulation

Setting the Agenda:

# voters	1	1	1
	b	a	c
	d	b	a
	c	d	b
	a	c	d

The order: 1. *a* vs. *b*; 2. the winner vs. *c*; 3. the winner vs. *d* elects *d*

The Danger of Manipulation

Setting the Agenda:

# voters	1	1	1
	b	a	c
	d	b	a
	c	d	b
	a	c	d

The order: 1. a vs. b; 2. a vs. c; 3. the winner vs. d elects d

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The Danger of Manipulation

Setting the Agenda:

# voters	1	1	1
	b	a	c
	d	b	a
	c	d	b
	a	c	d

The order: 1. a vs. b; 2. a vs. c; 3. c vs. d elects d, but **everyone** prefers b to d.

The Danger of Manipulation

"Insincere Voting":

# voters	3	3	1
	a	b	c
	b	a	a
	c	c	b

The Danger of Manipulation

"Insincere Voting":

# voters	3	3	1
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	b	a	a
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BC will elect *a* with 10 points (*b* gets 9 points and *c* gets 2 points).

The Danger of Manipulation

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	b	c	a
	c	a	b

BC will elect *a* with 10 points (*b* gets 9 points and *c* gets 2 points), but the middle group can be insincere and make *b* the winner

The Danger of Manipulation

"Failure of IIA":

# voters	3	2	2
	a	b	c
	b	c	a
	c	a	b

The Danger of Manipulation

"Failure of IIA":

# voters	3	2	2
	a	b	c
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	c	a	b

The BC ranking is: a (8) $>$ b (7) $>$ c (6)

The Danger of Manipulation

"Failure of IIA":

# voters	3	2	2
	a	b	c
	b	c	x
	c	x	a
	x	a	b

The BC ranking is: a (8) $>$ b (7) $>$ c (6)
Add a new (undesirable) candidate x.

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	c	x	a
	x	a	b

The BC ranking is: a (8) $>$ b (7) $>$ c (6)
Add a new (undesirable) candidate x.
The new BC ranking is: c (13) $>$ b (12) $>$ a (11) $>$ x (6)

Conclusions

- Many different types of voting methods: Plurality, Plurality with runoff, AV, BC, Hare system (STV), Copeland, Dodgson, Condorcet, etc.
- Many different dimensions to compare the procedures.
- No voting methods is perfect....

Thank You!

ai.stanford.edu/∼epacuit/lmh

Next Week: Michel Balinski

Next^2 Week: Steven Brams (Thursday)

Next^3 Week: Manipulability?