Logical Methods in the Humanities: Voting Theory

Stanford University
ai.stanford.edu/~epacuit/lmh

Fall, 2008

Topics

Main Question: Given a group of people faced with some decision, how should a central authority combine the individual opinions so as to best reflect the "will of the group"?

Topics

Main Question: Given a group of people faced with some decision, how should a central authority combine the individual opinions so as to best reflect the "will of the group"?

Typical Examples:

- Electing government officials
- Department meetings
- Deciding where to go to dinner with friends
- ....

Reflecting the will of the people

Reflecting the will of the people

- **Pareto Optimality**: If outcome a is unanimously preferred to outcome b, then b should not be the social choice.

Reflecting the will of the people

- **Pareto Optimality**: If outcome a is unanimously preferred to outcome b, then b should not be the social choice.
- **Anonymity**: The names of the voters do not matter (if two voters change votes, then the outcome is unaffected)

Reflecting the will of the people

- **Pareto Optimality**: If outcome a is unanimously preferred to outcome b, then b should not be the social choice.
- **Anonymity**: The names of the voters do not matter (if two voters change votes, then the outcome is unaffected)
- **Neutrality**: The names of the candidates, or options, do not matter (if two candidate are exchanged in every ranking, then the outcome changes accordingly)

Reflecting the will of the people

- **Pareto Optimality**: If outcome a is unanimously preferred to outcome b, then b should not be the social choice.
- **Anonymity**: The names of the voters do not matter (if two voters change votes, then the outcome is unaffected)
- **Neutrality**: The names of the candidates, or options, do not matter (if two candidate are exchanged in every ranking, then the outcome changes accordingly)
- **Monotonicity**: Moving up in the rankings is always better

Reflecting the will of the people

What about *majority voting*?

Reflecting the will of the people

What about *majority voting*?

If there are only **two** options, then majority voting is the "best" procedure.

Reflecting the will of the people: Majority Voting

Suppose that there are n individuals and two alternatives x and y

Reflecting the will of the people: Majority Voting

Suppose that there are n individuals and two alternatives x and y

For each $i \leq n$ there is a variable $D_i \in \{-1, 0, 1\}$ where

$$D = \begin{cases} -1 & \text{if} \quad y \text{ is preferred} \\ 0 & \text{if} \quad i \text{ is indifferent between } x \text{ and } y \\ 1 & \text{if} \quad x \text{ is preferred} \end{cases}$$

A **group decision function** is a map $f : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$

Reflecting the will of the people: Majority Voting

A group decision function f is

- **Decisive** if it is a total function

Reflecting the will of the people: Majority Voting

A group decision function f is

- **Decisive** if it is a total function
- **Symmetric** if $f(D_1, \ldots, D_n) = f(D_{j(1)}, \ldots, D_{j(n)})$ for all permutations j. I.e., f is symmetric in all of its arguments.

Reflecting the will of the people: Majority Voting

A group decision function f is

- **Decisive** if it is a total function
- **Symmetric** if $f(D_1, \ldots, D_n) = f(D_{j(1)}, \ldots, D_{j(n)})$ for all permutations j. I.e., f is symmetric in all of its arguments.
- **Neutral** if $f(-D_1, \ldots, -D_n) = -f(D_1, \ldots, D_n)$

Reflecting the will of the people: Majority Voting

A group decision function f is

- **Decisive** if it is a total function
- **Symmetric** if $f(D_1, \ldots, D_n) = f(D_{j(1)}, \ldots, D_{j(n)})$ for all permutations j. I.e., f is symmetric in all of its arguments.
- **Neutral** if $f(-D_1, \ldots, -D_n) = -f(D_1, \ldots, D_n)$
- **Positively Responsive** if $f(D_1, \ldots, D_n) = 0$ or 1, and $D'_i = D_i$ for all $i \neq i_0$, and $D'_{i_0} > D_{i_0}$, then $f(D'_1, \ldots, D'_n) = 1$

Reflecting the will of the people: Majority Voting

May's Theorem A group decision function is the method of simple majority decision if and only if it is decisive, symmetric, neutral and positively responsive

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

Generalizing May's Theorem

In May's Theorem, the agents are making a single binary choice between two alternatives. What about more general situations?

Generalizing May's Theorem

In May's Theorem, the agents are making a single binary choice between two alternatives. What about more general situations?

- Agents choose between between more than two alternatives.
- There are multiple interconnected propositions on which simultaneous decisions are to be made.

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

Brams and Fishburn. *Voting Procedures*. Handbook of Social Choice and Welfare (2002).

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

a is the simple majority winner.

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

But a **stronger** majority ranks *a* last.

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

Condorcet Winner: c beats each candidate in a pairwise comparisons.

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

Condorcet Winner: c beats each candidate in a pairwise comparisons.

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

Condorcet Winner: c beats each candidate in a pairwise comparisons.

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

Condorcet Winner: c beats each candidate in a pairwise comparisons.

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

Condorcet: c beats each candidate in a pairwise comparisons.

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

Borda: Take into account the *entire* ordering: all voters rank b and c either first, second or third.

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

Borda: Take into account the *entire* ordering: all voters rank b and c either first, second or third.

Reflecting the will of the people

# voters	3	5	7	6
	a	a	b	c
	b	c	d	b
	c	b	c	d
	d	d	a	a

Borda: Take into account the *entire* ordering: b best reflects the will of the people!

Main Question: Given a group of people faced with some decision, how should a central authority combine the individual opinions so as to best reflect the "will of the group"?

Many different answers to this question!

Main Question: Given a group of people faced with some decision, how should a central authority combine the individual opinions so as to best reflect the "will of the group"?

Many different answers to this question!

How should we compare the different methods?

Arrow's Theorem

Let X be a finite set of objects with *at least three elements*.

Assume each agent has a transitive and complete preference over X.

A **social welfare function** maps tuple of preferences over X to a preference over X.

Arrow's Theorem

Let X be a finite set of objects with *at least three elements*.

Assume each agent has a transitive and complete preference over X.

A **social welfare function** maps tuple of preferences over X to a preference over X.

Arrow's Theorem A social welfare function is a dictatorship iff it respects transitivity, is unanimous and satisfies independence of irrelevant alternatives.

K. Arrow. *Social Choice and Individual Values*. 1951.

Manipulation

It has long been noted that a voter can achieve a preferred election outcome by misrepresenting his or her actual preferences.

Manipulation

It has long been noted that a voter can achieve a preferred election outcome by misrepresenting his or her actual preferences.

C.L. Dodgson refers to a voters tendency to

> *"adopt a principle of voting which makes it a game of skill than a real test of the wishes of the elector."*

Manipulation

It has long been noted that a voter can achieve a preferred election outcome by misrepresenting his or her actual preferences.

C.L. Dodgson refers to a voters tendency to

> *"adopt a principle of voting which makes it a game of skill than a real test of the wishes of the elector."*

and that in his opinion

> *"it would be better for elections to be decided according to the wishes of the majority than of those who happen to be more skilled at the game."*

Manipulation

It has long been noted that a voter can achieve a preferred election outcome by misrepresenting his or her actual preferences.

C.L. Dodgson refers to a voters tendency to

> *"adopt a principle of voting which makes it a game of skill than a real test of the wishes of the elector."*

and that in his opinion

> *"it would be better for elections to be decided according to the wishes of the majority than of those who happen to be more skilled at the game."*

(Taken from A. Taylor *Social Choice and the Mathematics of Manipulation* who took it from D. Black *A Theory of Committees and Elections* who took it from Dodgson.)

The Gibbard-Satterthwaite Theorem

Gibbard-Satterthwaite Theorem There must be situations where it 'profits' a voter to vote *strategically*, i.e., not according to his or her *actual preference*.

The Gibbard-Satterthwaite Theorem

Gibbard-Satterthwaite Theorem There must be situations where it 'profits' a voter to vote *strategically*, i.e., not according to his or her *actual preference*.

Under suitable conditions,

1. If P denotes the actual preference ordering of voter i,
2. and $\vec{Y}$ denotes the profile consisting of the preference orderings of all the other voters,
3. and S the aggregation rule,

Then the theorem says that there must exist P, Y, P' such that $S(P', Y) >_P S(P, Y)$.

A. Gibbard. *Manipulation of Voting Schemes: A General Result*. Econometrica, 1973.

M. Satterthwaite. *Strategy-Proofness and Arrow's Conditions*. Journal of Economic Theory (1975).

The Logic of Group Decisions

Fundamental Problem: groups are inconsistent!

The Logic of Group Decisions: The Doctrinal "Paradox" (Kornhauser and Sager 1993)

P: a valid contract was in place
Q: there was a breach of contract
R: the court is required to find the defendant liable.

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

The Logic of Group Decisions: The Doctrinal "Paradox" (Kornhauser and Sager 1993)

Should we accept R?

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

The Logic of Group Decisions: The Doctrinal "Paradox" (Kornhauser and Sager 1993)

Should we accept R? No, a simple majority votes no.

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

The Logic of Group Decisions: The Doctrinal "Paradox" (Kornhauser and Sager 1993)

Should we accept R? Yes, a majority votes yes for P and Q and $(P \wedge Q) \leftrightarrow R$ is a legal doctrine.

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

Theorem (List and Pettit, 2001) There exists no judgement aggregation function generating complete, consistent and deductively closed collective sets of judgements which satisfies Universal Domain, Anonymity and Systematicity.

`personal.lse.ac.uk/LIST/doctrinalparadox.htm`

Plan for this Quarter

1. Introduction
2. Arrow's Theorem
3. Manipulation and the Gibbard-Satterthwaite Theorem
4. Voting Procedures (approval, Borda count, plurality, plurality with runoff) and Paradoxes (Condorcet paradox, no-show paradox, agenda manipulation)
5. Sen's Theorem and Generalizations of Arrow's Theorem
6. Judgement Aggregation, Domain Conditions, ...
7. Plus four speakers

A reader is available at a discounted price with the main material we will cover.

Thank You!

ai.stanford.edu/∼epacuit/lmh