# Judgement Aggregation

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# The Logic of Group Decisions

**Fundamental Problem:** groups are inconsistent!

# The Logic of Group Decisions: The Doctrinal "Paradox" (Kornhauser and Sager 1993)

$p$: a valid contract was in place
$q$: there was a breach of contract
$r$: the court is required to find the defendant liable.

| | $p$ | $q$ | $(p \wedge q) \leftrightarrow r$ | $r$ |
|---|---|---|---|---|
| 1 | yes | yes | yes | yes |
| 2 | yes | no | yes | no |
| 3 | no | yes | yes | no |

# The Logic of Group Decisions: The Doctrinal "Paradox" (Kornhauser and Sager 1993)

Should we accept $r$?

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Should we accept $r$? No, a simple majority votes no.

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Should we accept $r$? Yes, a majority votes yes for $p$ and $q$ and $(p \wedge q) \leftrightarrow r$ is a legal doctrine.

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**Conclusion**: Groups are inconsistent, difference between 'premise-based' and 'conclusion-based' decision making, ...

# The Judgement Aggregation Model: The Propositions

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*Aside*: We actually need

1. $\{p, \neg p\}$ are inconsistent
2. all subsets of a consistent set are consistent
3. $\emptyset$ is consistent and each $S \subseteq \mathcal{L}$ has a consistent maximal extension (not needed in all cases)

# The Judgement Aggregation Model: The Propositions

**Definition** A set $Y \subseteq \mathcal{L}$ is **minimally inconsistent** if it is inconsistent and every proper subset $X \subsetneq Y$ is consistent.

# The Judgement Aggregation Model: The Agenda

**Definition** The **agenda** is a non-empty set $X \subseteq \mathcal{L}$, interpreted as the set of propositions on which judgments are made, with $X$ is a union of proposition-negation pairs $\{p, \neg p\}$.

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**Example**: In the discursive dilemma:
$X = \{a, \neg a, b, \neg b, a \rightarrow b, \neg(a \rightarrow b)\}$.

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**Definition**: Given an agenda $X$, each individual $i$'s judgement set is a subset $A_i \subseteq X$.

**Rationality Assumptions**:

1. $A_i$ is **consistent**
2. $A_i$ is **complete**, if for each $p \in X$, either $p \in A_i$ or $\neg p \in A_i$

# The Judgement Aggregation Model: Aggregation Rules

Let $X$ be an agenda, $N = \{1, \ldots, n\}$ a set of voters, a **profile** is a tuple $(A_i, \ldots, A_n)$ where each $A_i$ is a judgement set. An **aggregation function** is a map from profiles to judgment sets. I.e., $F(A_1, \ldots, A_n)$ is a judgement set.

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**Examples**:

- **Propositionwise majority voting**: for each $(A_1, \ldots, A_n)$,

$$F(A_1, \ldots, A_n) = \{p \in X \mid |\{i \mid p \in A_i\}| \geq |\{i \mid p \notin A_i\}|\}$$

- **Dictator of** $i$: $F(A_1, \ldots, A_n) = A_i$
- **Reverse Dictator of** $i$: $F(A_1, \ldots, A_n) = \{\neg p \mid p \in A_i\}$

# The Judgement Aggregation Model: Aggregation Rules

**Universal Domain**: The domain of $F$ is the set of all possible profiles of consistent and complete judgement sets.

**Collective Rationality**: $F$ generates consistent and complete collective judgment sets.

## The Judgement Aggregation Model: Aggregation Rules

**Unanimity**: For all profiles $(A_1, \ldots, A_n)$ if $p \in A_i$ for each $i$ then $p \in F(A_1, \ldots, A_n)$

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**Independence**: For any $p \in X$ and all $(A_1, \ldots, A_n)$ and $(A_1^*, \ldots, A_n^*)$ in the domain of $F$,

$$\text{if [for all } i \in N,\ p \in A_i \text{ iff } p \in A_i^*]$$
$$\text{then } [p \in F(A_1, \ldots, A_n) \text{ iff } p \in F(A_1^*, \ldots A_n^*)\ ].$$

# The Judgement Aggregation Model: Aggregation Rules

**Systematicity**: For any $p, q \in X$ and all $(A_1, \ldots, A_n)$ and $(A^*_1, \ldots, A^*_n)$ in the domain of $F$,

$$\text{if [for all } i \in N,\ p \in A_i \text{ iff } q \in A^*_i]$$
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**Monotonicity**: For any $p \in X$ and all $(A_1, \ldots A_i, \ldots, A_n)$ and $(A_1, \ldots, A_i^*, \ldots, A_n)$ in the domain of $F$,

$$\text{if } [p \notin A_i,\ p \in A_i^* \text{ and } p \in F(A_1, \ldots, A_i, \ldots A_n)] \\ \text{then } [p \in F(A_1, \ldots, A_i^*, \ldots A_n)].$$

# The Judgement Aggregation Model: Aggregation Rules

**Anonymity**: If $(A_1, \ldots, A_n)$ and $(A_1^*, \ldots, A_n^*)$ are permutations of each other, then

$$F(A_1, \ldots, A_n) = F(A_1^*, \ldots, A_n^*)$$

**Non-dictatorship**: There exists no $i \in N$ such that, for any profile $(A_1, \ldots, A_n)$, $F(A_1, \ldots, A_n) = A_i$

# Baseline Result

**Theorem (List and Pettit, 2001)** If $X \subseteq \{a, b, a \wedge b\}$, there exists no aggregation rule satisfying universal domain, collective rationality, systematicity and anonymity.

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See `personal.lse.ac.uk/LIST/doctrinalparadox.htm` for many generalizations!

# Agenda Richness

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**Definition** An agenda $X$ is **minimally connected** if

1. it has a minimal inconsistent subset $Y \subseteq X$ with $|Y| \geq 3$
2. it has a minimal inconsistent subset $Y \subseteq X$ such that

$$Y - Z \cup \{\neg z \mid z \in Z\} \text{ is consistent}$$

   for some subset $Z \subseteq Y$ of even size.

**Theorem (Dietrich and List, 2007)** For a minimally connected agenda $X$, an aggregation rule $F$ satisfies universal domain, collective rationality, systematicity and the unanimity principle iff it is a dictatorship.

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*Pause for proof*

# Characterization Result

**Even-Number-Negation Property**: The agenda $X$ has a minimal inconsistent subset $Y \subseteq X$ such that $Y - Z \cup \{\neg z \mid z \in Z\}$ is consistent for some subset $Z \subseteq Y$ of even size.

**Median Property**: All minimally inconsistent subsets of the agenda $X$ contain exactly two propositions.

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**Median Property**: All minimally inconsistent subsets of the agenda $X$ contain exactly two propositions.

**Theorem (Dietrich and List, 2007)** There exists **regular**, systematic and non-dictatorial aggregation rules on the agenda $X$ iff $X$ satisfies the median property or violates the even-number-negation property.

**regular** means collectively rational, universal domain and unanimity

# Characterization Result

**Theorem (Nehring and Puppe 2006)** There exists regular, monotonic, systematic and non-dictatorial aggregation rules on the agenda $X$ iff $X$ has the median property.

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**Total Blockedness**: Say $p$ conditionally entails $q$ if $p \neq \neg q$ and there is a minimally inconsistent subset $Y \subseteq X$ such that $p, \neg q \in Y$.

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$X$ is **totally blocked** if for any pair $p, q \in X$ there is a sequence $p = p_1, \ldots, p_m = q$ where each $p_{i-1}$ conditionally entails $p_i$.

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**Theorem (Nehring and Puppe 2006)** There exists regular, monotonic, independent and non-dictatorial aggregation rules on the agenda $X$ iff $X$ is not totally blocked.

# Many Variants!

Christian List and Clemens Puppe. *Judgement Aggregation: A Survey*. 2007.

# Sen's Liberal Paradox

Two members of a small society Lewd and Prude each have a personal copy of *Lady Chatterley's Lover*, consider

## Sen's Liberal Paradox

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$l$: Lewd reads the book;
$p$: Prude reads the book;
$l \rightarrow p$: If Lewd reads the book, then so does Prude.

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Prude desires to not read the book, and that Lewd not read it either, but in case Lewd does read the book, Prude wants to read the book to be informed about the dangerous material Lewd has read.

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*So, society must be inconsistent!*

# Individual Rights

Call an individual $i$ decisive on a set $Y \subseteq X$ if any proposition in $Y$ is collectively accepted iff it is accepted by $i$, formally for each $(A_1, \ldots, A_n)$

$$F(A_1, \ldots, A_n) \cap Y = A_i \cap Y$$

**Minimal Rights** There exist (at least) two individuals who are each decisive an (at least) on proposition-negation pair $\{p, \neg p\} \subseteq X$.

# Individual Rights: Impossibility Theorem

**Theorem** If (and only if) the agenda is connected, there exists no aggregation function that satisfies universal domain, minimal rights and the unanimity principle.

Franz Dietrich and Christian List. *A Liberal Paradox for Judgment Aggregation*. Forthcoming.

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- Relations with database merging (Pigozzi, Konieczny)
- Bayesian,...

Thank You!

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