Universal axiomatizations of plane geometries in languages without relation symbols

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- Since the existence geometric objects is usually ensured by means of a geometric construction, it is natural to ask whether the existence quantifiers cannot be removed in these axioms, not by means of Skolemization, but by removing all relation symbols, and reformulating the entire theory in terms of operation symbols with a precise geometric meaning.
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- Language without predicate symbols, with two quaternary predicate and three individual constants.
- One could say that the operation symbols themselves stand for Geminus's *postulates* (which ask for the production, of something not yet given), whereas the axioms stand for his *axioms* (offering insight into the validity of certain relationships that hold between given notions).
- Geometry becomes a *quantifier-free* story about some geometric operations, all statements it makes amounting to the (in)equality of points, connected by the usual logical connectives, interpreted classically. Such axiomatizations will be called *constructive*.

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- E. Engeler (1968): algorithmic logic, containing only Boolean combinations of halting-formulas for flow-charts (that may contain loops but not recursive calls).
- The only aspect of Euclidean geometry over the field of real numbers this logic detects, beside the elementary geometry of the construction operations in its language, is the Archimedean axiom.
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- why several geometries that had been previously axiomatized classically were naturally occurring and not artificial;
- that simple geometries, such as plane Euclidean geometry, are full of very simple to formulate open questions;
- axiom systems for geometry that are absolutely simplest, by being statements about at most 4 variables, with only ternary operation symbols in the language.
- that the interesting ontological questions that arise are not those regarding (in)dependence of axioms, but rather those regarding the (in)dependence of the operation symbols.

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Fragments of Euclidean geometry

- Ruler and compass geometry (no continuity beyond line-circle intersection).
- Euclidean geometry with free mobility, or the geometry of ruler and dividers (no line-circle intersection continuity).
- the geometry of restricted ruler (can draw line joining two points, but not the intersection point of two lines), set square, and dividers (no Pasch axiom, no Euclidean parallel postulate).
- *Gaußian planes*: L/K a quadratic extension of a field K of characteristic $\neq 2$; $\{1, \sigma\}$ its Galois group, $\mathfrak{G}(L, K) := \langle L, \equiv \rangle$, with $xy \equiv uv$ iff $\|x - y\| = \|u - v\|$, with $\|x\| = x\sigma(x)$, for x, y, u, $v \in L$ (no free mobility, no order; however, Euclidean parallel postulate holds):.
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The most rudimentary fragment of Euclidean geometry

- rectangular planes of Karzel and Stanik (1979) (same metric structure as that of metric-Euclidean planes, but perpendicular lines no longer need intersect).

Constructive axiom systems for Euclidean geometry of ruler and compass

- In terms of T, U, H, $T(abc)$ being the point obtained by transporting ab on the ray $\overrightarrow{ac}$, $U(abc)$ being the circumcentre of triangle abc, and $H(abc)$ being one vertex of the right triangle with ac as hypotenuse, and b as foot of the perpendicular from $H(abc)$, in case a, b, c are three collinear points with b between a and c.
- Is T needed?.

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- In terms of T and U.
- Both are needed.

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Constructive axiom systems for Gaußian planes

- In terms of R, U; $R(abc) = d$ standing for 'd is the reflection of c to the line ab (if $a \neq b$; the reflection in a in the degenerate case $a = b$)', $U(abc)$ being the circumcentre of $\triangle abc$.
- Two sorted: in terms of φ, ι, γ, $\varphi(A, B)$ the line that passes through A and B if $A \neq B$; $\iota(g, h)$ the point of intersection of g and h; $\gamma(P, l)$ the perpendicular raised in P on l.
- Two sorted: in terms of φ, ι, γ', with $\gamma'(P, l)$ the perpendicular dropped from P to l, if P is not on l.
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Constructive axiom systems for Gaußian planes

- Two sorted: φ, ι and an operation corresponding to one aspect of the *restricted compass* (which may be used only to draw *uniquely determined* points of intersection of (i) circles and lines or (ii) circles and circles.

Metric-Euclidean planes

- P and F, with $P(abc)$ representing the image of c under the translation that maps a onto b, and $F(abc) = d$ standing for 'd is the foot of the perpendicular from c to the line ab (if $a \neq b$; a itself in the degenerate case $a = b$)'.
- Metric is Euclidean, but Euclidean parallel postulate need not hold.
- With $\sigma(ba) := P(abb)$, are F and σ sufficient to constructively axiomatize metric-Euclidean planes?
- Is P or σ needed at all? Can one do it with F alone?

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Rectangular planes

- P and R.
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Almost ordered metric-Euclidean planes with free mobility

- The Euclidean geometry of restricted ruler, set square, and segment-transporter constructions.
- one-sorted with T and M, operations corresponding to collapsible dividers and an instrument allowing the construction of the midpoint.
- $M(ab)$ the midpoint of ab; the point $T(abc)$ is as distant from a on the ray $\overrightarrow{ac}$ as b is from a, provided that $a \neq c \vee (a = c \wedge a = b)$, arbitrary, otherwise.
- Is M needed?

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Absolute geometry

- J' and T', J' is a quaternary segment-intersection predicate, $J'(abcd)$ being interpreted as the point of intersection of the segments ab and cd, provided that a and b are two distinct points that lie on different sides of the line cd, and c and d are two distinct points that lie on different sides of the line ab, and arbitrary otherwise; $T'(abc) = d$ if 'd is as distant from a on the ray $\overrightarrow{ca}$ as b is from a, provided that $a \neq c \vee (a = c \wedge a = b)$, and arbitrary, otherwise'.

Absolute geometry with *Lotschnittaxiom*

- *Lotschnittaxiom* (Bachmann (1964)): The perpendiculars raised on the sides of a right angle intersect.
- Do we need an additional geometric operation to constructively axiomatize the resulting geometry?
- No, as it turns out that the *Lotschnittaxiom* is equivalent to the statement that "In an isosceles triangle with half-right base angles, the altitude to the basis is less than the base itself."

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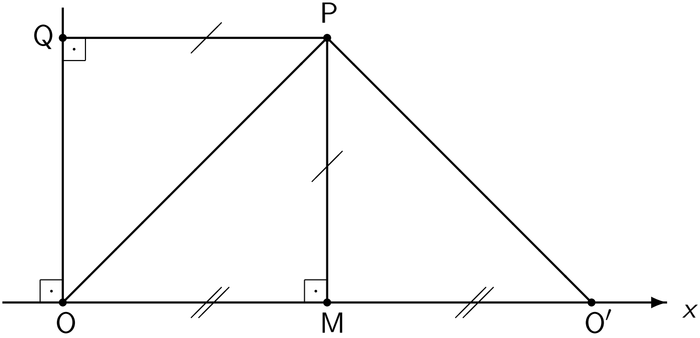

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Lagrange's axiom

- Motivated by the *principe de la raison suffisante*, which had been invoked by both Archimedes and Leibniz, Lagrange introduced on 3 February 1806 at the Institut de France the following axiom: "If a and b are two parallels from P to g, then the reflection of a in b is parallel to g as well", from which he deduced (assuming, as was customary at that time, the Archimedean axiom as well) the Euclidean parallel postulate.
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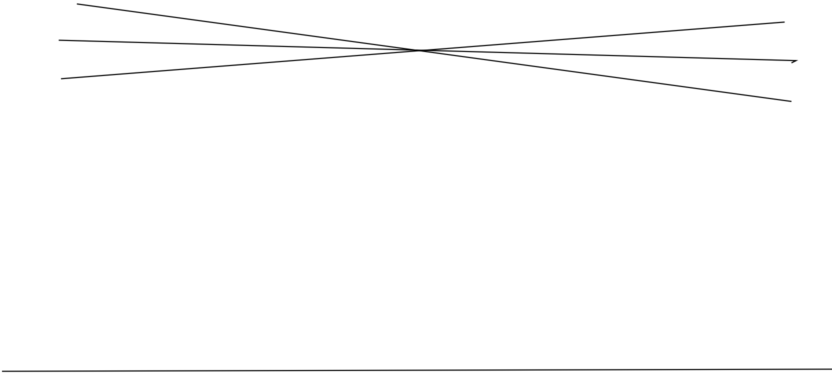

Figure: Lagrange's axiom

Aristotle's axiom

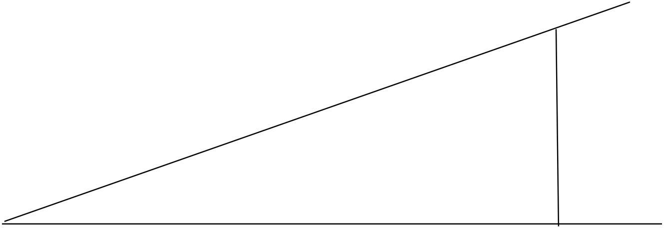

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Figure: Aristotle's's axiom

- Euclidean parallel postulate is equivalent to the conjunction of the *Lotscnittaxiom* and Aristotle's axiom

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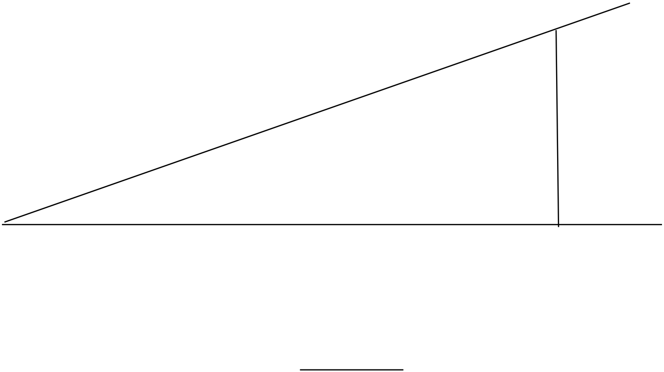

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Metric planes

- The theory of metric planes is that common substratum of Euclidean and non-Euclidean plane geometries that can be expressed in terms of incidence and orthogonality, where order, free mobility, and the intersection of non-orthogonal lines is ignored.
- The concept of a *metric plane*, one of the most remarkable concepts in the modern foundations of geometry, grew out of the work of Hessenberg, Hjelmslev, Reidemeister and A. Schmidt, and was provided with a simple group-theoretic axiomatics by Friedrich Bachmann in 1959.
- F and π, $\pi(abc)$ being interpreted as the fourth reflection point whenever a, b, c are collinear points with $a \neq b$ and $b \neq c$, an arbitrary point, otherwise.
- Is π needed? Could it be replaced by σ?

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- The theory of metric planes is that common substratum of Euclidean and non-Euclidean plane geometries that can be expressed in terms of incidence and orthogonality, where order, free mobility, and the intersection of non-orthogonal lines is ignored.
- The concept of a *metric plane*, one of the most remarkable concepts in the modern foundations of geometry, grew out of the work of Hessenberg, Hjelmslev, Reidemeister and A. Schmidt, and was provided with a simple group-theoretic axiomatics by Friedrich Bachmann in 1959.
- F and π, $\pi(abc)$ being interpreted as the fourth reflection point whenever a, b, c are collinear points with $a \neq b$ and $b \neq c$, an arbitrary point, otherwise.
- Is π needed? Could it be replaced by σ?

Plane hyperbolic geometry

- Can be constructively axiomatized both in languages with operation symbols whose geometric meaning is absolute, and thus Euclidean geometry can be axiomatized inside the language (such as T', J' and A, with $A(abc)$ representing the point on the ray $\overrightarrow{ac}$, whose distance from the line ab is congruent to the segment ab, provided that a, b, c are three non-collinear points, and an arbitrary point otherwise), and in languages with hyperbolic construction operations that have no Euclidean equivalent.
- two-sorted (point and lines): φ, ι, and $\pi_1(P, l) = g_1$, $\pi_2(P, l) = g_2$ may be read as 'g_1 and g_2 are the two limiting parallel lines from P to l' (provided that P is not on l, arbitrary lines, otherwise).

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Plane hyperbolic geometry

- (Klawitter (2003)): In L(a_0, a_1, I, ϵ_1, ϵ_2), where a_0, a_1, a_2 stand for three non-collinear points, with $\Pi(a_0a_1) = \pi/3$ ($\Pi(xy)$ standing here for the Lobachevsky function associating the angle of parallelism to the segment xy), $I(abcd)$ being the intersection point of lines ab and cd, whenever it exists, and ε_1 and ε_2, with $\varepsilon_i(abc) = d_i$ (for $i = 1, 2$) to be interpreted as 'd_1 and d_2 are two distinct points on line $\overline{ac}$ such that $ad_1 \equiv ad_2 \equiv ab$, provided that $a \neq c$, an arbitrary point, otherwise'.
- Also a one-sorted axiom system with ternary operations only, all axioms being universal statements about at most points.

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Klingenberg's generalized hyperbolic geometry

- Klingenberg (1954) axiomatized in the group-theoretical style of Bachmann a theory whose models are isomorphic to the *generalized Kleinian models* over arbitrary ordered fields K.
- Klingenberg's generalized hyperbolic geometry can be axiomatized in the bi-sorted (point, lines) first-order language $\mathrm{L}(A_0, A_1, A_2, A_3\ \varphi, F, \pi, \iota, \zeta)$, where A_0, A_1, A_2, A_3 stand for four points such that the lines $\varphi(A_0, A_1)$ and $\varphi(A_2, A_3)$ have neither a point nor a perpendicular in common, $\zeta(g, h)$ to be interpreted as 'the common perpendicular to g and h, provided that $g \neq h$ and that the common perpendicular exists, an arbitrary line, otherwise'.

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Hyperbolic geometry of set square, dividers, and restricted ruler

- What is the hyperbolic geometry of restricted ruler, set square and segment-transporter? A constructive axiom system for this natural fragment of hyperbolic geometry, can be stated in the one-sorted language with *points* as variables, $L(a_0, a_1, a_2, F, \tau)$, where τ is a segment transport operation.

Treffgeradenebenen

- Models whose *lines* are all the *Treffgeraden*, i. e. those lines in the two-dimensional Cartesian plane over K, with K a Pythagorean field, which intersect the unit circle in two points, and whose *points* are all points for which all lines of the plane that pass through them are *Treffgeraden*).
- Can be constructively axiomatized with A_0, A_1, A_2, φ, $\perp$, τ', λ_1, λ_2), where $\perp(P, g)$ stands for the foot of the perpendicular from P to g, $\{\tau'(A, B, C, g), \tau'(B, A, C, g)\}$ stand for the two points P on the line g, for which the segments CP and AB are congruent, provided that C lies on g, and $\{\lambda_1(A, B, C), \lambda_2(A, B, C)\}$ stand for the two lines that are hyperbolically parallel to $\varphi(A, B)$ and perpendicular to $\varphi(A, C)$, if A, B, C are three noncollinear points, and the lines $\varphi(A, B)$ and $\varphi(A, C)$ are not orthogonal.

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How do we know whether two lines are parallel or not?

- In Klingenberg planes as well as in *Treffgeradenebenen*, we have a metric criterion for the hyperbolic parallelism of two lines. It is Bergau's criterion
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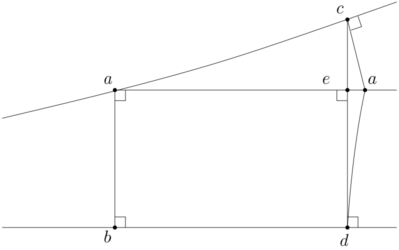

Figure: Bergau's criterion

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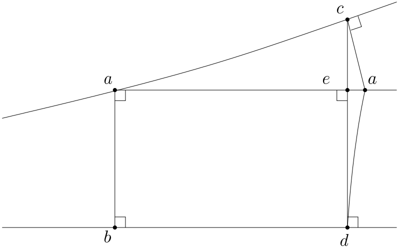
- 

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What is the hyperbolic geometry of midpoint and point-reflection constructions?

- In 2004, I've shown that plane hyperbolic geometry (as well as Klingenberg planes) can be axiomatized by means of $\forall\exists$-axioms in terms of the midpoint operation M (or of the point reflection operation σ).
- No axiom systems for the universal M-theory or the universal σ-theory or the universal $\{M, \sigma\}$-theory are known. Nor is it known whether these are finitely axiomatizable. In the Euclidean case, these theories are finitely axiomatizable and have a very simple structure.

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