Modal Language and Truth

Kai F. Wehmeier

Center for the Advancement of Logic, its Philosophy, History & Applications (C-ALPHA)
Department of Logic and Philosophy of Science
Department of Philosophy
University of California, Irvine

31 May 2015

The Actuality Modal Language

Inductive definition of the AML-formulas:

- $\phi ::= p \mid \neg\phi \mid (\phi \wedge \phi) \mid \Box\phi \mid \mathrm{A}\phi$

Models $\mathfrak{M} = (W, w^*, (p^{\mathfrak{M}})_{p \in \mathrm{SL}})$, where

- $w^* \in W$
- $p^{\mathfrak{M}} \subseteq W$ for each sentence letter $p \in \mathrm{SL}$

Recursive definition of $\mathfrak{M} \Vdash_w \phi$ ("ϕ is true at w in $\mathfrak{M}$"):

- $\mathfrak{M} \Vdash_w p$ iff $w \in p^{\mathfrak{M}}$
- $\mathfrak{M} \Vdash_w \neg\phi$ iff $\mathfrak{M} \nVdash_w \phi$
- $\mathfrak{M} \Vdash_w (\phi \wedge \psi)$ iff $\mathfrak{M} \Vdash_w \phi$ and $\mathfrak{M} \Vdash_w \psi$
- $\mathfrak{M} \Vdash_w \Box\phi$ iff for all $v \in W$, $\mathfrak{M} \Vdash_v \phi$
- $\mathfrak{M} \Vdash_w \mathrm{A}\phi$ iff $\mathfrak{M} \Vdash_{w^*} \phi$

The First-Order Languages $\mathscr{L}_{@}$, $\mathscr{L}_{\mathsf{A}}$, and $\mathscr{L}^{x}_{\mathsf{A}}$

Let $\mathscr{L}_{@}$ be the first-order language whose signature is given by

- an individual constant @
- a unary predicate symbol p for each sentence letter $p \in \mathrm{SL}$

AML-models $\mathfrak{M} = (W, w^*, (p^{\mathfrak{M}})_{p \in \mathrm{SL}})$ are first-order models for $\mathscr{L}_{@}$.

Let $\mathscr{L}_{\mathsf{A}}$ result from $\mathscr{L}_{@}$ by recasting @ as a generalized quantifier A.

Inductive definition of the $\mathscr{L}_{\mathsf{A}}$-formulas:

- $\phi ::= px \,|\, \neg\phi \,|\, (\phi \wedge \phi) \,|\, \forall x \phi \,|\, \mathsf{A}x\phi$

Semantics for quantifier A:

- $\mathfrak{M} \models_{\sigma} \mathsf{A}x\phi \;\Leftrightarrow\; \mathfrak{M} \models_{\sigma^{w^*}_{x}} \phi$

Let $\mathscr{L}^{x}_{\mathsf{A}}$ be the one-variable fragment of $\mathscr{L}_{\mathsf{A}}$ with respect to x.

Write $\mathfrak{M} \models_{w} \phi$ instead of $\mathfrak{M} \models_{\{\langle x, w \rangle\}} \phi$.

AML is $\mathscr{L}_{\mathsf{A}}^{x}$

Translate back and forth between AML and $\mathscr{L}_{\mathsf{A}}^{x}$ according to

$$\begin{array}{ccc} p & \rightleftharpoons & px \\ \mathsf{A} & \rightleftharpoons & \mathsf{A}x \\ \Box & \rightleftharpoons & \forall x \end{array}$$

E.g. $[(p \wedge \mathsf{A}\neg q) \to \Box(\mathsf{A}p \wedge q)] \rightleftharpoons (px \wedge \mathsf{A}x\neg qx) \to \forall x(\mathsf{A}xpx \wedge qx)$.

The translations are 1-1 and onto, and respect the semantics:

$$\text{If } \phi \in \text{AML}, \psi \in \mathscr{L}_{\mathsf{A}}^{x}, \text{ and } \phi \rightleftharpoons \psi, \text{ then}$$

$$\mathfrak{M} \Vdash_w \phi \quad \Leftrightarrow \quad \mathfrak{M} \models_w \psi.$$

Thus $(\text{AML}, \Vdash_w)$ and $(\mathscr{L}_{\mathsf{A}}^{x}, \models_w)$ are trivial notational variants.

The Case of Non-Universal Accessibility Relations

$\mathfrak{M} = (W, w^*, (R_w)_{w \in W}, (p^{\mathfrak{M}})_{p \in \mathsf{SL}})$ with $R_w \subseteq W$.

Inductive definition of $\mathscr{L}_{\mathsf{A}/R}$-formulas:

- $\phi ::= px \,|\, \neg\phi \,|\, (\phi \wedge \phi) \,|\, {}^{y}\forall x\phi \,|\, \mathsf{A}x\phi$

Satisfaction clause for ${}^{y}\forall x\phi$:

- $\mathfrak{M} \models_\sigma {}^{y}\forall x\phi$ iff for all $w \in R_{\sigma(y)}$, $\mathfrak{M} \models_{\sigma^w_x} \phi$.

Let $\mathscr{L}^x_{\mathsf{A}/R}$ be the one-variable fragment of $\mathscr{L}_{\mathsf{A}/R}$.

Translate back and forth between AML and $\mathscr{L}^x_{\mathsf{A}/R}$ according to

$$\begin{array}{ccc} p & \rightleftharpoons & px \\ \mathsf{A} & \rightleftharpoons & \mathsf{A}x \\ \Box & \rightleftharpoons & {}^{x}\forall x \end{array}$$

Then, if $\phi \rightleftharpoons \psi$, $\mathfrak{M} \Vdash_w \phi$ iff $\mathfrak{M} \models_w \psi$.

Thus $(\mathrm{AML}, \Vdash_w)$ and $(\mathscr{L}^x_{\mathsf{A}/R}, \models_w)$ are trivial notational variants.

Modal and Non-Modal Language

Truth in a Model

Natural Language

What's at stake for Philosophy?

Tarskian Truth for First-Order Languages

Tarski's Observation:

For sentences (closed formulas) ϕ, the following are equivalent:

(1) $\mathfrak{M} \models_\sigma \phi$

(2) $\mathfrak{M} \models_\tau \phi$ for all assignments τ

(3) $\mathfrak{M} \models_\tau \phi$ for some assignment τ

Briefly: For sentences, satisfaction is independent of assignments.

Define: Sentence ϕ is T-true in $\mathfrak{M}$, $\mathfrak{M} \models^T \phi$, if (1)–(3) hold.

Define: Sentence ϕ is T-false in $\mathfrak{M}$, $\mathfrak{M} \not\models^T \phi$, if (1)–(3) fail.

In particular, this applies to $\mathscr{L}_\mathsf{A}$ and $\mathscr{L}_\mathsf{A}^x$, hence to AML.

Tarskian Truth for AML

What corresponds to $\mathscr{L}^x_{\mathsf{A}}$-sentencehood in AML?

$\phi \in$ AML is operator-controlled if every sentence letter occurrence in ϕ lies within the scope of $\Box$ or of A.

E.g. $\mathsf{A}(\neg p \wedge \Box q)$ is OC; $(\neg p \wedge \Box q)$ is not OC.

By Tarski and notational variance, TFAE for OC formulas ϕ:

(i) ϕ is true at all worlds of $\mathfrak{M}$

(ii) ϕ is true at some worlds of $\mathfrak{M}$

(iii) ϕ is true at the actual world of $\mathfrak{M}$

Define: OCF ϕ is T-true in $\mathfrak{M}$, $\mathfrak{M} \Vdash^T \phi$, if (i)–(iii) hold.

Define: OCF ϕ is T-false in $\mathfrak{M}$, $\mathfrak{M} \nVdash^T \phi$, if (i)–(iii) fail.

NB: With non-universal accessibility relations, operator-control is not enough; we need A-control (because ${}^x\forall x$ introduces free x).

The Modal Orthodoxy: Postsemantics

Orthodox definition of truth in $\mathfrak{M}$ proceeds via a "postsemantics":

- $\mathfrak{M} \Vdash^K \phi \;:\Leftrightarrow\; \mathfrak{M} \Vdash_{w^*} \phi$.

With this postsemantics, AML is intensional:

Let $\mathfrak{M} = (W, w^*, (p^{\mathfrak{M}})_{p \in \mathrm{SL}})$ be a model with more than one world.

Suppose $p^{\mathfrak{M}} = W$ and $q^{\mathfrak{M}} = \{w^*\}$.

Then $\mathfrak{M} \Vdash^K p$ and $\mathfrak{M} \Vdash^K q$, because $\mathfrak{M} \Vdash_{w^*} p$ and $\mathfrak{M} \Vdash_{w^*} q$.

Also, $\mathfrak{M} \Vdash^K \Box p$ and $\mathfrak{M} \nVdash^K \Box q$.

So p and q have the same K-truth value.

But replacement of p by q in the K-true $\Box p$ leads to the K-false $\Box q$.

Hence the context "$\Box$" is intensional in AML relative to K-truth.

T-Truth and Extensionality

Under Tarskian truth, first-order languages are extensional.

Hence so is AML under T-truth.

Revisit intensionality of AML under K-truth:

$\mathfrak{M} = (W, w^*, (p^{\mathfrak{M}})_{p \in \mathrm{SL}})$; $|W| \geq 2$; $p^{\mathfrak{M}} = W$ and $q^{\mathfrak{M}} = \{w^*\}$.

$\mathfrak{M} \Vdash^K p$ and $\mathfrak{M} \Vdash^K q$, but $\mathfrak{M} \Vdash^K \Box p$ and $\mathfrak{M} \nVdash^K \Box q$.

The Tarskian observes:

Neither p nor q is OC, so neither has a T-truth value in $\mathfrak{M}$.

$\mathfrak{M} \Vdash^T \mathrm{A}p$ and $\mathfrak{M} \Vdash^T \mathrm{A}q$, and $\mathfrak{M} \Vdash^T \Box p$ and $\mathfrak{M} \nVdash^T \Box q$.

The Contingency Problem

The orthodoxy embraces the following principle:

(E) If ϕ formalizes a truth-apt non-modal sentence *S*, then $\Box\phi$ formalizes the statement that *S* is necessary.

Together with Tarskian semantics, (E) implies that every truth is a necessary truth:

- Let the true non-modal sentence *S* be formalized by ϕ.
- Since *S* is true, so is ϕ, hence ϕ must be OC ("Tarskianism").
- Since ϕ is OC and true, it is true at all worlds.
- Since ϕ is true at all worlds, $\Box\phi$ is true.
- By (E), *S* is a necessary truth.

So the Tarskian must reject (E).

Tarskian Necessitation

The Tarskian replaces (E) with

(E′) If ϕ formalizes a truth-apt non-modal sentence S, then $\Box\phi^{-\mathsf{A}}$ formalizes the statement that S is necessary, where $\phi^{-\mathsf{A}}$ results from ϕ by deleting all occurrences of A.

Examples:

- nine is odd $\mathsf{A}p$
- no matter how things might have gone, nine would be odd $\Box p$
- it's raining $\mathsf{A}q$
- under certain circumstances, it would be raining $\Diamond q$

(E), (E′), and Surface Syntax

The orthodoxy is impressed by modal locutions such as:

(1) it is necessary that nine is odd

(2) it is possible that Nixon got Carswell through

(3) it must be the case that nine is odd

(4) it may be the case that Nixon got Carswell through

However, (1) and (2) behave differently in French:

(1F) il est necéssaire que neuf soit impair.

(2F) il est possible que Nixon ait sauvé Carswell.

Also, compare the more idiomatic versions of (3) and (4):

(3I) nine must be odd.

(4I) Nixon may have gotten Carswell through.

Finally, consider other modal locutions:

(5) no matter how things might have gone, nine would have been odd

(6) under certain circumstances, Nixon would've gotten Carswell through

(7) it is obligatory that someone introduce the speaker

(8) if I had bought a ticket, I would have won the lottery

A Sketch of a Tarskian Analysis of English

Sentence letters represent moodless English clauses.

E.g. $p \approx$ "nine be odd" and $q \approx$ "Parker chair the committee".

A represents indicative mood:

$p \mapsto \mathrm{A}p$ corresponds to "nine be odd" $\mapsto$ "nine is odd".

$\Box$ represents a modal locution plus "subjunctive" mood.

Depending on the application, the modal locution might be

- "no matter how things might have gone," and subjunctive mood may manifest as "would":
  "nine be odd" $\mapsto$ "no matter how. . . , nine would be odd"
- "it is obligatory that," and subjunctive mood may manifest as, well, subjunctive mood:
  "Quentin confess" $\mapsto$ "it is obligatory that Quentin confess"

NB: Non-OCFs instantiate to sentences without a truth value.

Modal and Non-Modal Language

Truth in a Model

Natural Language

What's at stake for Philosophy?

Contingent Logical Truths?

Zalta (1988): AML contains contingent logical truths: $\mathsf{A}p \leftrightarrow p$.

1. Fact: $\mathsf{A}p \leftrightarrow p$ is true at the actual world in all models.
2. Assumption: Truth is K-truth, i.e. truth at the actual world.
3. From 1 and 2: $\mathsf{A}p \leftrightarrow p$ is true in all models.
4. Assumption: Logical truth is truth in all models.
5. From 3 and 4: $\mathsf{A}p \leftrightarrow p$ is logically true.
6. Fact: There are models $\mathfrak{M}$ in which $\Box(\mathsf{A}p \leftrightarrow p)$ is false.
7. From 6 and (E): Relative to such $\mathfrak{M}$, $\mathsf{A}p \leftrightarrow p$ is a contingent truth.

As Tarskians, we reject 2: Truth is T-truth, not K-truth.

Hence 3 fails: $\mathsf{A}p \leftrightarrow p$ isn't OC, so not true (or false) in any model.

Accordingly, 5 and 7 fail, too.

NB: Tarskians also reject (E).

END