Whence Necessity?

Tom Donaldson, June 2015

1. Introduction

Some putative necessary truths:

- $(\alpha \vee \neg\alpha)$
- It is wrong to break a promise.
- Hesperus=Phosphorous
- 7+5=12
- Every grain of table salt is a small crystal of NaCl.

How do we account for the necessity of these propositions?

It may be that some necessities are beyond explanation. Here are some plausible examples:

- For all *x*, *y* and *z*, if *x* is between *y* and *z* then *x* is between *z* and *y*.
- Nothing has mass 10g and also mass 20g.

(Wang, 'From Combinatorialism to Primitivism')

However, I take it that we will not wish to say that all necessities are *brute* in this way.

The plan for today:

- I will use facts about cardinal numbers as a case study.
- I will not today consider logical truths.
- I will begin by consider ‘The Semantic Strategy’. (§2)
- Then I will consider ‘The Ground-Theoretic Strategy’. (§3)
- I’ll finish (§4) by suggesting that we want to ‘Mix and Match’ – that is, we might want to use different strategies in different cases.

2. The Semantic Strategy

#FlamingoSpecies

#WelshCity

Hume's Principle:

$$\forall X \forall Y(\#X=\#Y \leftrightarrow \exists W\, X \sim_W Y)$$

'$X \sim_W Y$' abbreviates the conjunction of these four conditions:

$$\forall x(Xx \rightarrow \forall y_1(Yy_1 \rightarrow \forall y_2(Yy_2 \rightarrow (Wxy_1 \wedge Wxy_2 \rightarrow y_1=y_2))))$$
$$\forall y(Yy \rightarrow \forall x_1(Xx_1 \rightarrow \forall x_2(Xx_2 \rightarrow (Wx_1y \wedge Wx_2y \rightarrow x_1=x_2))))$$
$$\forall x(Xx \rightarrow \exists y(Yy \wedge Wxy))$$
$$\forall y(Yy \rightarrow \exists x(Xx \wedge Wxy))$$

For example, Hume's Principle tells us that the following two propositions have the same truth value:

$\#\mathit{FlamingoSpecies} = \#\mathit{WelshCity}$

$\exists \mathrm{W}\ \mathit{FlamingoSpecies} \sim_{\mathrm{W}} \mathit{WelshCity}$

Greater Flamingo	⟵⟶	Bangor
Lesser Flamingo	⟵⟶	Cardiff
Chilean Flamingo	⟵⟶	Newport
Andean Flamingo	⟵⟶	St Davids
James’s Flamingo	⟵⟶	Swansea
American Flamingo	⟵⟶	St Asaph

Hume's Principle:

$$\forall X \forall Y(\#X=\#Y \leftrightarrow \exists W\ X \sim_W Y)$$

0	abbreviates	$\#\langle \lambda x : x \neq x \rangle$
1	abbreviates	$\#\langle \lambda x : x = 0 \rangle$
2	abbreviates	$\#\langle \lambda x : x = 0 \lor x = 1 \rangle$
(etc.)		

Sum(x, y, z) abbreviates

$$\exists X\, \exists Y\, (\neg \exists w (Xw \land Yw) \land x = \#X \land y = \#Y \land z = \#\langle \lambda w : Xw \lor Yw \rangle)$$

‘Julius invented the zip fastener.’

(Evans, ‘Reference and Contingency’)

Plausibly,

- the proposition that Julius invented the zip fastener is analytic…
- … but contingent.

3. The Ground-Theoretic Strategy

“Tom’s trousers fell down because his braces snapped.”

Cause: The snapping of Tom’s braces.

Effect: The descent of Tom’s trousers.

(a) Tom's outfit was ugly, because his shirt didn’t match his pants.

(b) This is a grain of table salt, because it is a small crystal of NaCl.

(c) This piece of paper is square because it is flat, has four straight sides of equal length, and the angles at its corners are equal.

(a′)
GrounDED fact: [Tom's outfit was ugly]
GrounDING fact: [Tom's shirt did not match Tom's pants]

(b′)
GrounDED fact: [This is a grain of table salt]
GrounDING fact: [This is a small crystal of NaCl]

(c′)
GrounDED fact: [This piece of paper is square]
GrounDING facts:
[This piece of paper is flat],
[This piece of paper has four equal straight sides]
[The angles at the corners of this piece of paper are equal.]

Some claims about grounding from the philosophical literature:

- Mental facts are grounded by physical facts.
- Facts about what is legal or illegal are grounded by facts about the actions of officials, and the institutions in which they operate.
- Normative facts are grounded by natural facts.
- Aesthetic facts are grounded by facts about ideal or typical observers.

(etc.)

I represent the relation of total ground using a leftward pointing arrow '$\leftarrow$', thus:

$$F_1 \leftarrow \{F_2, F_3, F_4\}$$

For any facts F_0 and F_1, $F_0 \mathrel{|\!\!\leftarrow} F_1$ iff there is some set Γ such that $F_0 \leftarrow \Gamma$ and $F_1 \in \Gamma$.

The “pure logic” of ground:

- The relation of partial ground is a strict partial order.
- The relation of partial ground is well-founded.

The “impure logic” of ground:

- If $(\alpha \wedge \beta)$, then $[\alpha \wedge \beta] \leftarrow \{[\alpha], [\beta]\}$.
- If α, then $[\alpha \vee \beta] \leftarrow [\alpha]$ and $[\beta \vee \alpha] \leftarrow [\alpha]$.
- If $\varphi(a)$, the $[\exists x\ \varphi(x)] \leftarrow \{[\varphi(a)]\}$.
- If $\forall x\ \varphi(x)$, then $[\forall x\ \varphi(x)]$ is totally grounded by the set of facts of the form $[\varphi(x)]$ (+ a totality fact?).

(etc.)

Particularly important for us is the following claim, ‘grounding necessitarianism’:

If $([\varphi] \leftarrow \{[\psi_1], [\psi_2], [\psi_3], \ldots\})$, then $\Box((\psi_1 \wedge \psi_2 \wedge \ldots) \rightarrow \varphi)$.

The Schwartzkopff-Rosen Principle:

For all X and Y, if $\#X=\#Y$, then $[\#X=\#Y] \leftarrow \{[\exists W\ X \sim_W Y]\}$.

For all X and Y, if $\#X\neq\#Y$, then $[\#X\neq\#Y] \leftarrow \{[\neg\exists W\ X \sim_W Y]\}$.

For example:

$[\#\mathit{WelshCity} = \#\mathit{FlamingoSpecies}]$
$\leftarrow \{[\exists W\ \mathit{WelshCity} \sim_W \mathit{FlamingoSpecies}]\}$.

Now consider:

$$[Sum(1, 1, 2)]$$

Given the above definitions, this fact is just:

$$[\exists X\ \exists Y\ (\neg\exists w(Xw \wedge Yw) \wedge\ \#\langle\lambda x : x{=}\#\langle\lambda x : x{\neq}x\rangle\rangle{=}\#X \wedge \#\langle\lambda x : x{=}\#\langle\lambda x : x{\neq}x\rangle\rangle{=}\#Y \wedge \#\langle\lambda x : x{=}\#\langle\lambda x : x{\neq}x\rangle \vee x{=}\#\langle\lambda x : x{=}\#\langle\lambda x : x{\neq}x\rangle\rangle\rangle{=}\#\langle\lambda w : Xw \vee Yw\rangle)]$$

A problem: The Schwartzkopff-Rosen Principle, when combined with various standard claims about the ‘impure logic of ground’ leads to a regress of ground:

$$[1=1] ⇤ [\neg 1=0] ⇤ [0=0] ⇤ \ldots$$

We should not assume that [the grounding relation] is well founded. That is a substantive question. It may be natural to suppose that every fact ultimately depends on an array of basic facts, which in turn depend on nothing. But it might turn out, for all we know, that the facts about atoms are grounded in facts about quarks and electrons, which are in turn grounded in facts about 'hyperquarks' and 'hyperelectrons' and so on *ad infinitum*. So we should leave it open that there might be an infinite chain of facts [p]←[q]←[r]←…

(Rosen, 'Metaphysical Dependence: Grounding and Reduction')

$$[1=1] \mid\!\!\leftarrow [\neg 1=0] \mid\!\!\leftarrow [0=0] \mid\!\!\leftarrow \ldots$$

The Schwartzkopff-Rosen Principle:

For all X and Y, if #X=#Y, then [#X=#Y] $\leftarrow$ {[$\exists$W X $\sim_W$ Y]}.

For all X and Y, if #X≠#Y, then [#X≠#Y] $\leftarrow$ {[$\neg\exists$W X $\sim_W$ Y]}.

'X $\sim_W$ Y' abbreviates the conjunction of the following four conditions:

$$\forall x(Xx \rightarrow \forall y_1(Yy_1 \rightarrow \forall y_2(Yy_2 \rightarrow (Wxy_1 \wedge Wxy_2 \rightarrow y_1{=}y_2))))$$

$$\forall y(Yy \rightarrow \forall x_1(Xx_1 \rightarrow \forall x_2(Xx_2 \rightarrow (Wx_1y \wedge Wx_2y \rightarrow x_1{=}x_2))))$$

$$\forall x(Xx \rightarrow \exists y(Yy \wedge Wxy))$$

$$\forall y(Yy \rightarrow \exists x(Xx \wedge Wxy))$$

Response: Adopt Alex Skiles' account of what grounds facts involving *restricted* quantification. Here is Skiles' proposal:

If $(\forall x\colon \varphi(x))\ \psi(x)$, then $[(\forall x\colon \varphi(x))\ \psi(x)]$ is totally grounded by the set $\{[\psi(a)], [\psi(b)], [\psi(c)], \ldots\}$ (where a, b, c, … are all the things x such that $\psi(x)$).

For example, for Skiles the fact that every twentieth-century Dutch monarch is female is totally grounded by:

{[*Female*(Wilhelmina)], [*Female*(Juliana)], [*Female*(Beatrix)]}

(Skiles, 'Against Grounding Necessitarianism')

The new version of The Schwartzkopff-Rosen Principle:

For all X and Y, if #X=#Y, then

$$[\#X=\#Y] \leftarrow \{[(\exists W{:}\mathrm{Simple}_{X,Y}(W))\ X \approx_W Y]\}.$$

For all X and Y, if #X≠#Y, then

$$[\#X\neq\#Y] \leftarrow \{[\neg(\exists W{:}\mathrm{Simple}_{X,Y}(W)\ X \approx_W Y]\}.$$

‘$X \approx_W Y$’ abbreviates the conjunction of the following four conditions:

$$(\forall x{:}Fx)(\forall y_1{:}Gy)(\forall y_2{:}Gy)(Rxy_1 \wedge Rxy_2 \rightarrow y_1{=}y_2)$$

$$(\forall y{:}Gy)(\forall x_1{:}Fx)(\forall x_2{:}Fx)(Rx_1y \wedge Rx_2y \rightarrow x_1{=}x_2)$$

$$(\forall x{:}Fx)(\exists y{:}Gy)\ Rxy$$

$$(\forall y{:}Gy)(\exists x{:}Fx)\ Rxy$$

It is a consequence of the revised version of The Schwartzkopff-Rosen Principle that all facts of the form [x=x] and [$\neg$x=y] where x, y $\in \{0, 1, 2, \ldots \aleph_0\}$ are zero-grounded, so the regress problem is solved.

"Trivialism is the view that true sentences of pure mathematics have trivial truth-conditions … According to the trivialist, nothing is required of the world in order for the truth-conditions of a mathematical truth to be satisfied"

(Rayo, 'Towards a Trivialist Account of Mathematics')

A problem: Skiles' proposal conflicts with grounding necessitarianism.

(a) [Baldev promised to feed Oscar.]

(b) [Baldev is obliged to feed Oscar.]

“Enablers”:

- Baldev’s promise was not made under duress
- Baldev is capable of feeding Oscar.

The groundED fact: $[(\forall x: M(x))\ F(x)]$

The facts that do the groundING: [F(Wilhelmina)], [F(Juliana)] [F(Beatrix)]

The enabling fact:

$[(\forall x: M(x))(x\text{=Wilhelmina} \vee x\text{=Juliana} \vee x\text{=Beatrix})]$

Grounding Necessitarianism (Naïve Version):

If $([\varphi] \leftarrow \{[\psi_1], [\psi_2], [\psi_3], \ldots\})$, then $\Box((\psi_1 \wedge \psi_2 \wedge \ldots) \rightarrow \varphi)$.

Grounding Necessitarianism (New Version):

Suppose $([\varphi] \leftarrow \{[\psi_1], [\psi_2], [\psi_3], \ldots\})$, with enablers $\beta_1, \beta_2, \beta_3, \ldots$ Then $\Box((\psi_1 \wedge \beta_1 \wedge \psi_2 \wedge \beta_2 \wedge \ldots) \rightarrow \varphi)$.

Let $F = \langle \lambda x\text{: } x=1 \rangle$, $R=\langle \lambda xy\text{: } x=1 \wedge y=1 \rangle$. Then:

$$
\begin{array}{ll}
[1=1] & \\
\leftarrowtail & [(\exists R\text{:Simple}(W))\ F \approxeq_W F] \\
\leftarrowtail & [F \approxeq_R F] \\
= & [\text{Function}(R) \wedge \text{Injective}(R) \wedge \text{Total}(R) \wedge \text{Onto}(R)] \\
\leftarrowtail & [\text{Total}(R)] \\
= & [(\forall x\text{:}Fx)(\exists y\text{:}Fy)\ Rxy] \\
\leftarrowtail & [(\exists y\text{:}Fy)\ R1y] \\
\leftarrowtail & [R11] \\
= & [\langle \lambda xy\text{: } x=1 \wedge y=1 \rangle 11] \\
\leftarrowtail & [1=1 \wedge 1=1] \\
\leftarrowtail & [1=1]
\end{array}
$$

Tentative conclusions:

- The grounding strategy may yet turn out to be successful, however …
- … we may need to give up on some widely-accepted and *prima facie* plausible general claims about grounding.
- For example, we may have to give up on the claim that the relation of partial ground is acyclic.
- Perhaps more importantly, the ‘naïve version’ of grounding necessitarianism should be replaced with some more sophisticated claim.
- More work is required to explain what grounds facts about cardinal numbers greater than $\aleph_0$.

4. Mix and Match

One might wish to use the semantic strategy in one instance and the grounding strategy in another.

E.g. One might be convinced, on epistemological grounds, to use the semantic strategy in the case of arithmetic. Consideration of Moore’s open question argument might convince you that the semantic strategy can’t work in the case of ethics. So in that case one might choose to use the grounding strategy.

In fact, this ‘Mix and Match’ approach is *very* rare.

Why?

I shall suppose that facts are structured entities built up from worldly items – objects, relations, connectives, quantifiers etc. – in roughly the sense in which sentences are built up from words. … Facts are individuated by their constituents and the manner of their composition. This yields a very fine-grained notion. If [p] and [q] are distinct, then [$p\vee\neg p$] is distinct from [$q\vee\neg q$]. And this is as it should be. The fact that [$p\vee\neg p$] might be grounded by [p]. But [p] cannot possibly ground [$q\vee\neg q$] except in special cases.

(Rosen, 'Metaphysical Dependence: Grounding and Reduction')

So for Rosen, these are distinct:

$[\#F=\#G]$

$[\exists W\ F \sim_W G]$

However, in section §64 of the *Grundlagen,* Frege implies the sentences “$\#F=\#G$” and “$\exists W\ F \sim_W G$” have the same “content” *(inhalt)* although this content is “carved up” in different ways in the two cases.

My summer project: solve this problem!

I hope that the problem can be solved by introducing a theory of propositions with two characteristics:

(a) The proposition expressed by ‘$p \vee \neg p$’ may be distinct from the proposition expressed by ‘$q \vee \neg q$’.

(b) The proposition expressed by ‘$\#F = \#G$’ may be identical to the proposition expressed by ‘$\exists W \, F \sim_W G$’.