CSLI Workshop on Logic, Rationality and Intelligent Interaction
May 30–31, 2015, Stanford, CA, U.S.A.

Relevant contributions to modal logic [handout]

Katalin Bimbó
University of Alberta, Edmonton, Canada

Semantics. The minimal tense logic $\mathbf{K_t}$ is defined by the axioms (A0)–(A6) and the rules *detachment* and *necessitation*.

(A0, R1) A set of axioms for two-valued propositional logic with detachment.

(A1–2)	$\mathsf{G}(\mathcal{A} \wedge \mathcal{B}) \equiv (\mathsf{G}\mathcal{A} \wedge \mathsf{G}\mathcal{B})$	$\mathsf{P}(\mathcal{A} \vee \mathcal{B}) \equiv (\mathsf{P}\mathcal{A} \vee \mathsf{P}\mathcal{B})$
(A3–4)	$\mathsf{G}\mathcal{A} \supset \mathsf{GG}\mathcal{A}$	$\mathsf{PP}\mathcal{A} \supset \mathsf{P}\mathcal{A}$
(A5–6)	$\mathsf{PG}\mathcal{A} \supset \mathcal{A}$	$\mathsf{FH}\mathcal{A} \supset \mathcal{A}$
(R2–3)	$\vdash \mathcal{A} \;\Rightarrow\; \vdash \mathsf{G}\mathcal{A}$	$\vdash \mathcal{A} \;\Rightarrow\; \vdash \mathsf{H}\mathcal{A}$
(D1–2)	$\mathsf{F}\mathcal{A} \equiv_{\mathrm{df}} \neg\mathsf{G}\neg\mathcal{A}$	$\mathsf{H}\mathcal{A} \equiv_{\mathrm{df}} \neg\mathsf{P}\neg\mathcal{A}$

The *Boolean gaggle* $\mathfrak{G}_{\mathbf{K_t}}$ is defined as $\langle A; -, \wedge, \vee, f, h\rangle$, where (a1)–(a4) hold.

(a1) $\langle A; -, \wedge, \vee\rangle$ is a Boolean algebra;
(a2) $f(0) = 0, \quad h(1) = 1, \quad f(a \vee b) = f(a) \vee f(b), \quad h(a \wedge b) = h(a) \wedge h(b)$;
(a3) $h(a) \leq h(h(a)), \quad f(f(a)) \leq f(a)$;
(a4) $f(a) \leq b$ iff $a \leq h(b)$. [f and h are *abstractly residuated*.]

f and h form a *Galois connection* between $\langle A; \leq\rangle$ and $\langle A; \leq^{-1}\rangle$, where $\leq^{-1}$ is the converse of $\leq$. That is, (1) if $a \leq b$, then $f(b) \leq^{-1} f(a)$ and $a \leq^{-1} b$ implies $h(b) \leq h(a)$; (2) $a \leq h(f(a))$ and $a \leq^{-1} f(h(a))$.

Decidability. *Multiplicative–exponential linear logic* (**MELL**) is the following sequent calculus. (Γ, Δ, Θ and Λ are *multisets* of formulas.)

$$\mathcal{A} \vdash \mathcal{A}$$

$$\frac{\mathcal{A}; \mathcal{B}; \Gamma \vdash \Delta}{\mathcal{A} \otimes \mathcal{B}; \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta; \mathcal{A} \qquad \Theta \vdash \Lambda; \mathcal{B}}{\Gamma; \Theta \vdash \Delta; \Lambda; \mathcal{A} \otimes \mathcal{B}}$$

$$\frac{\mathcal{A}; \Gamma \vdash \Delta \qquad \mathcal{B}; \Theta \vdash \Lambda}{\mathcal{A} \parr \mathcal{B}; \Gamma; \Theta \vdash \Delta; \Lambda} \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}; \mathcal{B}}{\Gamma \vdash \Delta; \mathcal{A} \parr \mathcal{B}}$$

$$\frac{\Gamma \vdash \Delta; \mathcal{A}}{\mathcal{A}^{\perp}; \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta; \mathcal{A} \qquad \mathcal{B}; \Theta \vdash \Lambda}{\mathcal{A} \multimap \mathcal{B}; \Gamma; \Theta \vdash \Delta; \Lambda} \qquad \frac{\mathcal{A}; \Gamma \vdash \Delta; \mathcal{B}}{\Gamma \vdash \Delta; \mathcal{A} \multimap \mathcal{B}} \qquad \frac{\mathcal{A}; \Gamma \vdash \Delta}{\Gamma \vdash \Delta; \mathcal{A}^{\perp}}$$

$$\frac{\mathcal{A}; \Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta} \qquad \frac{!\Gamma \vdash ?\Delta; \mathcal{A}}{!\Gamma \vdash ?\Delta; !\mathcal{A}} \qquad \frac{\mathcal{A}; !\Gamma \vdash ?\Delta}{?\mathcal{A}; !\Gamma \vdash ?\Delta} \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}}{\Gamma \vdash \Delta; ?\mathcal{A}}$$

$$\frac{!\mathcal{A}; !\mathcal{A}; \Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta}\; {\scriptstyle !W\vdash} \qquad \frac{\Gamma \vdash \Delta; ?\mathcal{A}; ?\mathcal{A}}{\Gamma \vdash \Delta; ?\mathcal{A}}\; {\scriptstyle \vdash ?W} \qquad \frac{\Gamma \vdash \Delta}{!\mathcal{A}; \Gamma \vdash \Delta} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta; ?\mathcal{A}}$$

The *demi-relevant logic* **RL** is obtained by replacing $(!W\vdash)$ with $(W\vdash)$, and $(\vdash ?W)$ with $(\vdash W)$.

$$\frac{\mathcal{A}; \mathcal{A}; \Gamma \vdash \Delta}{\mathcal{A}; \Gamma \vdash \Delta}\; {\scriptstyle W\vdash} \qquad \frac{\Gamma \vdash \Delta; \mathcal{A}; \mathcal{A}}{\Gamma \vdash \Delta; \mathcal{A}}\; {\scriptstyle \vdash W}$$

Some relevant references and sources

Anderson, A. R., Belnap, N. D. & Dunn, J. M. (1992). *Entailment: The Logic of Relevance and Necessity*, Vol. II, Princeton University Press, Princeton, NJ.

Bimbó, K. (2007a). Functorial duality for ortholattices and De Morgan lattices, *Logica Universalis* **1**: 311–333.

Bimbó, K. (2007b). Relevance logics, *in* D. Jacquette (ed.), *Philosophy of Logic*, Vol. 5 of *Handbook of the Philosophy of Science* (D. Gabbay, P. Thagard and J. Woods, eds.), Elsevier (North-Holland), Amsterdam, pp. 723–789.

Bimbó, K. (2009). Dual gaggle semantics for entailment, *Notre Dame Journal of Formal Logic* **50**: 23–41.

Bimbó, K. (2014). The decidability of the intensional fragment of classical linear logic, *Theoretical Computer Science*. (29 pages, submitted on July 14th, 2014).

Bimbó, K. (2015). *Proof Theory: Sequent Calculi and Related Formalisms*, Discrete Mathematics and its Applications, CRC Press, Boca Raton, FL.

Bimbó, K. & Dunn, J. M. (2008). *Generalized Galois Logics. Relational Semantics of Nonclassical Logical Calculi*, Vol. 188 of *CSLI Lecture Notes*, CSLI Publications, Stanford, CA.

Bimbó, K. & Dunn, J. M. (2009). Symmetric generalized Galois logics, *Logica Universalis* **3**: 125–152.

Bimbó, K. & Dunn, J. M. (2012). New consecution calculi for $R^{t}_{\rightarrow}$, *Notre Dame Journal of Formal Logic* **53**: 491–509.

Bimbó, K. & Dunn, J. M. (2013). On the decidability of implicational ticket entailment, *Journal of Symbolic Logic* **78**: 214–236.

Bimbó, K. & Dunn, J. M. (2014–2015). Modalities in lattice-R. (ms., 37 pages).

Dunn, J. M. (1976). A Kripke-style semantics for R-mingle using a binary accessibility relation, *Studia Logica* **35**: 163–172.

Dunn, J. M. (1986). Relevance logic and entailment, *in* D. Gabbay & F. Guenthner (eds.), *Handbook of Philosophical Logic*, 1st edn, Vol. 3, D. Reidel, Dordrecht, pp. 117–224.

Dunn, J. M. (1991). Gaggle theory: An abstraction of Galois connections and residuation with applications to negation, implication, and various logical operators, *in* J. van Eijck (ed.), *Logics in AI: European Workshop JELIA '90*, number 478 in *Lecture Notes in Computer Science*, Springer, Berlin, pp. 31–51.

Dunn, J. M. (1993). Partial-gaggles applied to logics with restricted structural rules, *in* K. Došen & P. Schroeder-Heister (eds.), *Substructural Logics*, Clarendon, Oxford, UK, pp. 63–108.

Dunn, J. M. (1995a). Gaggle theory applied to intuitionistic, modal and relevance logic, *in* I. Max & W. Stelzner (eds.), *Logik und Mathematik. Frege-Kolloquium Jena*, W. de Gruyter, Berlin, pp. 335–368.

Dunn, J. M. (1995b). Positive modal logic, *Studia Logica* **55**: 301–317.

Dunn, J. M. (2001). A representation of relation algebras using Routley–Meyer frames, *in* C. A. Anderson & M. Zelëny (eds.), *Logic, Meaning and Computation. Essays in Memory of Alonzo Church*, Kluwer, Dordrecht, pp. 77–108.

Dunn, J. M. & Hardegree, G. M. (2001). *Algebraic Methods in Philosophical Logic*, Vol. 41 of *Oxford Logic Guides*, Oxford University Press, Oxford, UK.

Kripke, S. A. (1959). The problem of entailment, (abstract), *Journal of Symbolic Logic* **24**: 324.

Kripke, S. A. (1963). Semantical analysis of modal logic I. Normal modal propositional calculi, *Zeitschrift für mathematische Logik und Grundlagen der Mathematik* pp. 67–96.

Kripke, S. A. (1965a). Semantical analysis of intuitionistic logic I, *in* J. N. Crossley & M. A. E. Dummett (eds.), *Formal Systems and Recursive Functions. Proceedings of the Eighth Logic Colloquium*, North-Holland, Amsterdam, pp. 92–130.

Kripke, S. A. (1965b). Semantical analysis of modal logic II. Non-normal propositional calculi, *in* J. W. Addison, L. Henkin & A. Tarski (eds.), *The Theory of Models*, North-Holland, Amsterdam, pp. 206–220.

Routley, R. & Meyer, R. K. (1972). The semantics of entailment – II, *Journal of Philosophical Logic* **1**: 53–73.

Routley, R. & Meyer, R. K. (1973). The semantics of entailment, *in* H. Leblanc (ed.), *Truth, Syntax and Modality. Proceedings of the Temple University Conference on Alternative Semantics*, North-Holland, Amsterdam, pp. 199–243.